

The low-dimensional structures that tricategories form

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Abstract

We form tricategories and the homomorphisms between them into a bicategory. We then enrich this bicategory into an example of a three-dimensional structure called a *locally double bicategory*, this being a bicategory enriched in the monoidal 2-category of weak double categories. Finally, we show that every sufficiently well-behaved locally double bicategory gives rise to a tricategory, and thereby deduce the existence of a tricategory of tricategories.

1 Introduction

A major impetus behind many developments in 2-dimensional category theory has been the observation that, just as the fundamental concepts of set theory are categorical in nature, so the fundamental concepts of category theory are 2-categorical in nature. In other words, if one wishes to study categories “in the small” – as mathematical entities in their own right rather than as universes of discourse – then a profitable way of doing this is by studying the 2-categorical properties of **Cat**, the 2-category of all categories.¹

Once one moves from the study of categories to the study of (possibly weak) n -categories, it is very natural to generalise the above maxim, and to assert that *the fundamental concepts of n -category theory are $(n+1)$ -categorical in nature*. This is a

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¹Here, and elsewhere, we will adopt a common-sense attitude to set-theoretic issues, assuming a sufficient supply of Grothendieck universes and leaving it to the reader to qualify entities with suitable constraints on their size.

profitable thing to do: for example, consider the coherence theorem for bicategories [16], which in its simplest form states that

Every bicategory is biequivalent to a 2-category.

A priori, this is merely a statement about individual bicategories; but we may also read it as a statement about the tricategory **Bicat** of all bicategories and cells between them, since “biequivalent” is shorthand for “internally biequivalent in the tricategory **Bicat**.”² Thus another way of stating the above would be to say that the 2-categories are biequivalence-dense in **Bicat**.

This maxim permeates almost all research in higher-dimensional category theory, and so we draw attention to here, not in order to point out where we might use it, but rather where we might *not* use it. For instance, consider once again the coherence theorem for bicategories. We may restate it slightly more tightly as:

*Every bicategory is biequivalent to a 2-category
via an identity-on-objects biequivalence.*

The restriction to identity-on-objects biequivalences affords us an interesting simplification, since, as pointed out in [15], we can express such a biequivalence as a mere *equivalence* in a suitable 2-category, which we denote by **Bicat**₂. The 0-cells of **Bicat**₂ are the bicategories; the 1-cells are the homomorphisms between them; and the 2-cells are *icons*, of which we will see more in the next section: for now let us just say that they are special oplax natural transformations, which exist only between homomorphisms which agree on 0-cells.

With the help of this 2-category **Bicat**₂, the coherence theorem for bicategories can be made into a 2-categorical, rather than tricategorical, statement: namely, that the 2-categories are equivalence-dense in **Bicat**₂ (cf. Theorem 5.4 of [15]). This is a somewhat tighter result; moreover, the 2-category **Bicat**₂ is much simpler to work with than the tricategory **Bicat**. Thus we should revise our general maxim, and acknowledge that *some* of the fundamental concepts of *n*-category theory may be expressible using fewer than $(n + 1)$ dimensions.

Consequently, when we study *n*-categories, it may be useful to form them not only into an $(n + 1)$ -category, but also into suitable lower-dimensional structures, and it is the purpose of this paper to do so in the case where $n = 3$. We will construct both a bicategory of tricategories and a tricategory of tricategories: in both cases, the 2-cells are suitably scaled-up analogues of the bicategorical icons mentioned above. However, we will see that there is some information which the mere bicategory carries which the tricategory cannot, and thus we are led to consider a new kind of low-dimensional structure, which we call a *locally double bicategory*, that is rich enough to encode all the information from both the bicategory and tricategory of tricategories which we define.

²In practice, we will often use the more local definition of biequivalence, which says that \mathcal{B} is biequivalent to \mathcal{B}' if there exists a homomorphism $F: \mathcal{B} \rightarrow \mathcal{B}'$ which is surjective on objects up-to-equivalence, and which is locally an equivalence of categories; but as long as we assume the axiom of choice, the difference between the two definitions is merely one of presentation.

Notation. We follow [1] and [13] where it concerns 2- and bicategories: so in particular, our oplax natural transformations $\alpha: F \Rightarrow G$ have 2-cell components $\alpha_f: \alpha_B.Ff \Rightarrow Gf.\alpha_A$. We will tend to use either juxtaposition or the connective “.” to denote composition, relying on context to sort out precisely which sort of composition is intended. When it comes to tricategories, our primary references are [8] and [11], but with a preference for the “algebraic” presentation of the latter: though we will not use this algebraicity in any essential way.

We will also make use of *pasting diagrams* of 2-cells inside bicategories. Such diagrams are only well-defined up to a choice of bracketing of their boundary, and so we assume such a choice to have been made wherever necessary. Occasionally we will need to use similar pasting diagrams of 2-cells in a tricategory, and the same caveat holds, only more so: here, the diagram is only well-defined up to a choice of order in which the pasting should be performed; and again, we assume such a choice to have been made. We adopt one further convention regarding pasting diagrams. Suppose we are given a 2-cell $\alpha: h(gf) \Rightarrow h'(g'f')$ in a bicategory \mathcal{B} , thus:

$$\begin{array}{ccccc}
 & & X & \xrightarrow{g} & Y \\
 & f \nearrow & & & \searrow h \\
 W & & & \Downarrow \alpha & \\
 & f' \searrow & & & \nearrow h' \\
 & & X' & \xrightarrow{g'} & Y' \\
 & & & & Z,
 \end{array}$$

together with a homomorphism of bicategories $F: \mathcal{B} \rightarrow \mathcal{C}$. Applying F to α yields a 2-cell $F(h(gf)) \Rightarrow F(h'(g'f'))$ of \mathcal{C} , but frequently, we will be more interested in the 2-cell

$$\begin{array}{ccccc}
 & & FX & \xrightarrow{Fg} & FY \\
 & Ff \nearrow & & & \searrow Fh \\
 FW & & & \Downarrow & \\
 & Ff' \searrow & & & \nearrow Fh' \\
 & & FX' & \xrightarrow{Fg'} & FY' \\
 & & & & FZ;
 \end{array}$$

obtained by pasting $F\alpha$ with suitable coherence constraints for the homomorphism F : and we will consistently denote the 2-cell obtained in this way by $\overline{F\alpha}$.

2 A bicategory of tricategories

We begin by describing the lowest-dimensional structure into which tricategories and their homomorphisms can form themselves. At first, one might think that this would be a category; but unfortunately, composition of trihomomorphisms fails to be associative on the nose, as it requires one to compose 1-cells in a hom-bicategory,

which is itself not an associative operation. Consequently, the best we can hope for is a *bicategory* of tricategories, which we will denote by **Tricat**₂.

The simplest such bicategory would have trihomomorphisms as its 1-cells and *blips* as its 2-cells. According to [8], blips are very degenerate tritransformations which can only exist between two trihomomorphisms $F, G: \mathcal{S} \rightarrow \mathcal{T}$ which agree on 0-, 1-, 2- and 3-cells. Though one might think that this forces F and G to be the same, they can in fact differ with respect to certain pieces of coherence data, and a “blip” is the means by which one measures these differences.

However, if we are going to form a bicategory of tricategories, it may as well be the most general possible one; and so we will consider more general sorts of both 1- and 2-cells. Let us begin by looking at the 1-cells.

Definition 1. Let \mathcal{S} and \mathcal{T} be tricategories. A **lax homomorphism** $F: \mathcal{S} \rightarrow \mathcal{T}$ is a lax morphism of tricategories in the sense of [8], all of whose coherence 3-cells are invertible. Hence F consists in:

- A function $F: \text{ob } \mathcal{S} \rightarrow \text{ob } \mathcal{T}$;
- Homomorphisms of bicategories $F_{A,B}: \mathcal{S}(A, B) \rightarrow \mathcal{T}(FA, FB)$;
- 2-cells $\iota_A: I_{FA} \rightarrow FI_A$;
- 2-cells $\chi_{f,g}: Fg.Ff \Rightarrow F(gf)$, pseudo-natural in f and g ;
- Invertible modifications ω , δ and γ witnessing the coherence of the data for ι and χ ,

all subject to the axioms for a morphism of tricategories as found in [8].

The notion of lax homomorphism is a sensible one from many angles. We can compose lax homomorphisms just as we would compose homomorphisms of tricategories. If we are given a pair of monoidal bicategories [5] which we view as one-object tricategories, then the lax homomorphisms between them are the natural bicategorical generalisation of a lax monoidal functor (the *weak monoidal homomorphisms*, in the terminology of [5]). Lax homomorphisms from 1 into a tricategory \mathcal{T} classify *pseudomonads* in \mathcal{T} – that is, monads whose associativity and unit laws have been weakened to hold up to coherent isomorphism, and in a similar vein we may use lax homomorphisms to give a succinct definition of an *enriched bicategory* in the sense of [3, 14] – that is of a bicategory “enriched in a tricategory”, which is a one-dimension-higher version of a category enriched in a bicategory, which is in turn a generalisation of the familiar notion of a category enriched in a monoidal category. We shall see a little more of enriched bicategories in Section 3.

Let us now discuss the 2-cells of **Tricat**₂. The most informative precedent is the corresponding notion one dimension down: the *icons* of [15]. As we mentioned in the Introduction, these are certain degenerate oplax transformations which arise between homomorphisms of bicategories which agree on 0-cells. To be precise, given two such homomorphisms $F, G: \mathcal{B} \rightarrow \mathcal{C}$, an **icon** $\alpha: F \Rightarrow G$ is given by specifying for each 1-cell $f: A \rightarrow B$ of \mathcal{B} , a 2-cell $\alpha_f: Ff \Rightarrow Gf$ of \mathcal{C} such that:

- For each 2-cell $\sigma: f \Rightarrow g$ of \mathcal{B} , the following diagram commutes:

$$\begin{array}{ccc} Ff & \xRightarrow{\alpha_f} & Gf \\ F\sigma \Downarrow & & \Downarrow G\sigma \\ Fg & \xRightarrow{\alpha_g} & Gg; \end{array}$$

- For each object $A \in \mathcal{B}$, the following diagram commutes:

$$\begin{array}{ccc} \text{id}_{FA} & \xRightarrow{\cong} & F\text{id}_A \\ \parallel & & \Downarrow \alpha_{\text{id}_A} \\ \text{id}_{GA} & \xRightarrow{\cong} & G\text{id}_A; \end{array}$$

- For each pair of composable 1-cells $f: A \rightarrow B$, $g: B \rightarrow C$ in \mathcal{B} , the following diagram commutes:

$$\begin{array}{ccc} Fg.Ff & \xRightarrow{\cong} & F(gf) \\ \alpha_g.\alpha_f \Downarrow & & \Downarrow \alpha_{gf} \\ Gg.Gf & \xRightarrow{\cong} & G(gf), \end{array}$$

where the arrows labelled with \cong are expressing the pseudo-functoriality of F and G . Now, icons are precisely those oplax natural transformations whose components are all identities (hence the name: *identity component oplax natural transformation*), though the above description suppresses any mention of these identity components. Similarly, the 2-cells of \mathbf{Tricat}_2 we are about to describe can be seen as some sort of degenerate oplax tritransformation, with the degenerate data suppressed.³

So, let $F, G: \mathcal{S} \rightarrow \mathcal{T}$ be lax homomorphisms; then a 2-cell $\alpha: F \Rightarrow G$ of \mathbf{Tricat}_2 exists only if F and G agree on objects and 1-cells of \mathcal{S} , and is then given by the following data:

(TD1) For each pair of objects $A, B \in \mathcal{S}$, an icon

$$\alpha_{A,B}: F_{A,B} \Rightarrow G_{A,B}: \mathcal{S}(A, B) \rightarrow \mathcal{T}(FA, FB)$$

(and in particular, for each 2-cell $\theta: f \Rightarrow g$ of \mathcal{S} , a 3-cell of \mathcal{T} :

$$\begin{array}{ccc} Ff & \xRightarrow{F\theta} & Fg \\ \parallel & \Downarrow \alpha_\theta & \parallel \\ Gf & \xRightarrow{G\theta} & Gg \end{array});$$

³Though in order to make this precise, we would first have to define what an “oplax tritransformation” is.

(TD2) For each object A of \mathcal{S} , a 3-cell of \mathcal{T} :

$$\begin{array}{ccc} I_{FA} & \xRightarrow{\iota_A^F} & FI_A \\ \parallel & \Downarrow M_A^\alpha & \parallel \\ I_{GA} & \xRightarrow{\iota_A^G} & GI_A; \end{array}$$

(TD3) For each pair of composable 1-cells $f: A \rightarrow B, g: B \rightarrow C$ of \mathcal{S} , a 3-cell of \mathcal{T} :

$$\begin{array}{ccc} Fg.Ff & \xRightarrow{\chi_{f,g}^F} & F(gf) \\ \parallel & \Downarrow \Pi_{f,g}^\alpha & \parallel \\ Gg.Gf & \xRightarrow{\chi_{f,g}^G} & G(gf); \end{array}$$

subject to the following axioms:

(TA1) For each pair of 2-cells $\theta: f \Rightarrow g: B \rightarrow C$ and $\theta': f' \Rightarrow g': A \rightarrow B$ of \mathcal{S} , the following pasting equality holds:

$$\begin{array}{c} \begin{array}{ccccc} & Ff.Ff' & \xRightarrow{\chi^F} & F(ff') & \\ & \Downarrow \Pi^\alpha & & \searrow F(\theta\theta') & \\ Gf.Gf' & \xRightarrow{\chi^G} & G(ff') & & F(gg') \\ & \searrow G\theta.G\theta' & \Downarrow \chi^G & \searrow G(\theta\theta') & \\ & Gg.Gg' & \xRightarrow{\chi^G} & G(gg') & \end{array} \\ \\ = \begin{array}{ccccc} & Ff.Ff' & \xRightarrow{\chi^F} & F(ff') & \\ & \searrow F\theta.F\theta' & \Downarrow \chi^F & \searrow F(\theta\theta') & \\ Gf.Gf' & & Fg.Fg' & \xRightarrow{\chi^F} & F(gg') \\ & \searrow G\theta.G\theta' & \parallel & \searrow \Pi^\alpha & \\ & Gg.Gg' & \xRightarrow{\chi^G} & G(gg') & \end{array} \end{array}$$

(TA2) For each 1-cell $f: A \rightarrow B$ of \mathcal{S} , the following pasting equality holds:

$$\begin{array}{c}
\begin{array}{ccccc}
& & FI_B.Ff & \xrightarrow{\chi^F} & F(I_B.f) \\
& \nearrow \iota^F.1 & & & \searrow F\mathfrak{l} \\
I_{FB}.Ff & & & & Ff \\
\parallel & \searrow \mathfrak{l} & & & \parallel \\
I_{GB}.Gf & = & Ff & = & Ff & = & Gf \\
& \searrow \mathfrak{l} & & & \parallel & & \parallel \\
& & Gf & = & Gf & & Gf
\end{array} \\
\Downarrow \gamma^F \\
\begin{array}{ccccc}
& & FI_B.Ff & \xrightarrow{\chi^F} & F(I_B.f) \\
& \nearrow \iota^F.1 & & & \searrow F\mathfrak{l} \\
I_{FB}.Ff & & & & Ff \\
\parallel & \searrow M^{\alpha}.1 & & & \parallel \\
I_{GB}.Gf & \xrightarrow{\iota^G.1} & GI_B.Gf & \xrightarrow{\chi^G} & G(I_B.f) & \searrow G\mathfrak{l} & Ff \\
& & \parallel & & \parallel & & \parallel \\
& & & & Gf & & Gf
\end{array} \\
\Downarrow \gamma^G \\
\begin{array}{ccccc}
& & FI_B.Ff & \xrightarrow{\chi^F} & F(I_B.f) \\
& \nearrow \iota^F.1 & & & \searrow F\mathfrak{l} \\
I_{FB}.Ff & & & & Ff \\
\parallel & \searrow \mathfrak{l} & & & \parallel \\
I_{GB}.Gf & = & Ff & = & Ff & = & Gf \\
& \searrow \mathfrak{l} & & & \parallel & & \parallel \\
& & Gf & = & Gf & & Gf
\end{array}
\end{array}$$

(TA3) For each 1-cell $f: A \rightarrow B$ of \mathcal{S} , the following pasting equality holds:

$$\begin{array}{c}
\begin{array}{ccccc}
& & Ff.FI_B & \xrightarrow{\chi^F} & F(f.I_B) \\
& \nearrow 1.\iota^F & & & \searrow F\mathfrak{r} \\
Ff.I_{FB} & & & & Ff \\
\parallel & \searrow \mathfrak{r} & & & \parallel \\
Gf.I_{GB} & = & Ff & = & Ff & = & Gf \\
& \searrow \mathfrak{r} & & & \parallel & & \parallel \\
& & Gf & = & Gf & & Gf
\end{array} \\
\Downarrow \delta^F \\
\begin{array}{ccccc}
& & Ff.FI_B & \xrightarrow{\chi^F} & F(f.I_B) \\
& \nearrow 1.\iota^F & & & \searrow F\mathfrak{r} \\
Ff.I_{FB} & & & & Ff \\
\parallel & \searrow \mathfrak{r} & & & \parallel \\
Gf.I_{GB} & = & Ff & = & Ff & = & Gf \\
& \searrow \mathfrak{r} & & & \parallel & & \parallel \\
& & Gf & = & Gf & & Gf
\end{array}
\end{array}$$

$$\begin{array}{c}
= \\
\begin{array}{ccccc}
& & Ff.FI_B & \xrightarrow{\chi^F} & F(f.I_B) \\
& \nearrow 1.\iota^F & \parallel & \Downarrow \Pi^\alpha & \searrow F\tau \\
Ff.I_{FB} & \Downarrow 1.M^\alpha & Gf.GI_B & \xrightarrow{\chi^G} & G(f.I_B) & \Downarrow \alpha_\tau & Ff \\
\parallel & \nearrow 1.\iota^G & & \searrow G\tau & \parallel & & \\
Gf.I_{GB} & & & \Downarrow \delta^G & & & Gf. \\
& \searrow \tau & & & \nearrow & & \\
& Gf & \xlongequal{\quad} & Gf & & &
\end{array}
\end{array}$$

(TA4) For each triple f, g, h of composable 1-cells of \mathcal{S} , the following pasting equality holds:

$$\begin{array}{c}
\begin{array}{ccccc}
& & F(hg).Ff & \xrightarrow{\chi^F} & F((hg)f) \\
& \nearrow \chi^F.1 & & \searrow F\alpha & \\
(Fh.Fg).Ff & & & & F(h(gf)) \\
\parallel & \searrow \alpha & & \nearrow \chi^F & \parallel \\
(Gh.Gg).Gf & = & Fh.(Fg.Ff) & \xrightarrow{1.\chi^F} & Fh.F(gf) & \Downarrow \Pi^\alpha & G(h(gf)) \\
& \searrow \alpha & \parallel & \searrow 1.\Pi^\alpha & \parallel & \nearrow \chi^G & \\
& & Gh.(Gg.Gf) & \xrightarrow{1.\chi^G} & Gh.G(gf) & &
\end{array} \\
= \\
\begin{array}{ccccc}
& & F(hg).Ff & \xrightarrow{\chi^F} & F((hg)f) \\
& \nearrow \chi^F.1 & \parallel & \Downarrow \Pi^\alpha & \searrow F\alpha \\
(Fh.Fg).Ff & \Downarrow \Pi^\alpha.1 & G(hg).Gf & \xrightarrow{\chi^G} & G((hg)f) & \Downarrow \alpha_\alpha & F(h(gf)) \\
\parallel & \nearrow \chi^G.1 & & \searrow G\alpha & \parallel & & \\
(Gh.Gg).Gf & & & \Downarrow \omega^G & & & G(h(gf)). \\
& \searrow \alpha & & & \nearrow \chi^G & & \\
& Gh.(Gg.Gf) & \xrightarrow{1.\chi^G} & Gh.G(gf) & & &
\end{array}
\end{array}$$

When we reach Section 3, we will see that (TD3) and (TA1) express that the maps Π^α are the components of an $(\text{ob } \mathcal{S})^3$ -indexed family of “cubical modifications”

$$\begin{array}{ccc} F(-) \otimes F(?) & \xrightarrow{\chi^F} & F((-) \otimes (?)) \\ \alpha_{(-)} \otimes \alpha_{(?)} \downarrow & \Downarrow \Pi_{A,B,C}^\alpha & \downarrow \alpha_{((-) \otimes (?))} \\ G(-) \otimes G(?) & \xrightarrow[\chi_G]{} & G((-) \otimes (?)), \end{array}$$

where a “cubical modification” lives inside a square, bounded horizontally by pseudo-natural transformations and vertically by icons, and can be thought of as a modification between the oplax transformations obtained by composing up the two routes around this square.

Now, in order to make this collection of 0-, 1- and 2-cells into a bicategory, we have to give additional *data* – vertical composition of 2-cells, horizontal composition of 1- and 2-cells and associativity and unitality constraints – subject to additional *axioms* – the category axioms for vertical composition, the middle-four interchange axiom and the pentagon and triangle axioms for the associativity and unit constraints.

We start with the vertical structure: the identity 2-cell $\text{id}_F: F \Rightarrow F$ in **Tricat**₂ we take to be given by the following data:

$$(\text{id}_F)_{A,B} = \text{id}_{F_{A,B}}, \quad M_A^{\text{id}_F} = \text{id}_{\iota_A^F} \quad \text{and} \quad \Pi_{f,g}^{\text{id}_F} = \text{id}_{\chi_{f,g}^F}.$$

Each of the axioms (TA1)–(TA4) now expresses that something is equal to itself pasted together with some identity 3-cells, which is clear enough. Next, given 2-cells $\alpha: F \Rightarrow G$ and $\beta: G \Rightarrow H$ in **Tricat**₂, we take $\beta\alpha: F \Rightarrow H$ to be given by the following data:

$$(\beta\alpha)_{A,B} = \beta_{A,B} \cdot \alpha_{A,B}, \quad M_A^{\beta\alpha} = M_A^\beta \cdot M_A^\alpha \quad \text{and} \quad \Pi_{f,g}^{\beta\alpha} = \Pi_{f,g}^\beta \cdot \Pi_{f,g}^\alpha.$$

Each of the axioms (TA1)–(TA4) for this data follow from juxtaposing the corresponding axioms for α and β in a very straightforward manner. Moreover, because vertical composition of 3-cells in a tricategory is strictly associative and unital, so is the vertical composition of 2-cells in **Tricat**₂.

We turn now to the horizontal structure. Horizontal identities and composition for 1-cells are the identities and composition for lax homomorphisms as detailed in [14]; whilst given 2-cells $\alpha: F \Rightarrow F': \mathcal{S} \rightarrow \mathcal{T}$ and $\beta: G \Rightarrow G': \mathcal{T} \rightarrow \mathcal{U}$, their horizontal composite $\beta * \alpha: GF \Rightarrow G'F': \mathcal{S} \rightarrow \mathcal{U}$ is given by:

(TD1) $(\beta * \alpha)_{A,B} := \beta_{A,B} * \alpha_{A,B}$, where $*$ on the right-hand side is the horizontal composite of the underlying icons in the 2-category **Bicat**₂ of the Introduc-

tion. In particular, given a 2-cell $\theta: f \Rightarrow g$ of \mathcal{S} , we have

$$\begin{array}{c}
 GFf \xRightarrow{GF\theta} GFg \\
 \parallel \quad \Downarrow \beta_{F\theta} \quad \parallel \\
 (\beta * \alpha)_\theta = G'Ff \xRightarrow{G'F\theta} G'Fg \\
 \parallel \quad \Downarrow G'\alpha_\theta \quad \parallel \\
 G'F'f \xRightarrow{G'F'\theta} G'F'g;
 \end{array}$$

(TD2)

$$\begin{array}{c}
 I_{GFA} \xRightarrow{\iota^G} GI_{FA} \xRightarrow{G\iota^F} GF I_A \\
 \parallel \quad \Downarrow M_{FA}^\beta \quad \parallel \quad \Downarrow \beta_{\iota^F} \quad \parallel \\
 M_A^{\beta * \alpha} := I_{G'FA} \xRightarrow{\iota^{G'}} G' I_{FA} \xRightarrow{G'\iota^F} G' F I_A \\
 \parallel \quad = \quad \parallel \quad \Downarrow G' M_A^\alpha \quad \parallel \\
 I_{G'F'A} \xRightarrow{\iota^{G'}} G' I_{F'A} \xRightarrow{G'\iota^{F'}} G' F' I_A,
 \end{array}$$

(TD3)

$$\begin{array}{c}
 GFg.GFf \xRightarrow{\chi^G} G(Fg.Ff) \xRightarrow{G\chi^F} GF(gf) \\
 \parallel \quad \Downarrow \Pi_{Ff,Fg}^\beta \quad \parallel \quad \Downarrow \beta_{\chi^F} \quad \parallel \\
 \Pi_{f,g}^{\beta * \alpha} := G'Fg.G'Ff \xRightarrow{\chi^{G'}} G'(Fg.Ff) \xRightarrow{G'\chi^F} G'F(gf) \\
 \parallel \quad = \quad \parallel \quad \Downarrow G'\Pi_{f,g}^\alpha \quad \parallel \\
 G'F'g.G'F'f \xRightarrow{\chi^{G'}} G'(F'g.F'f) \xRightarrow{G'\chi^{F'}} G'F'(gf).
 \end{array}$$

We must check that these data satisfy (TA1)–(TA4). If we view the pasting equalities in these axioms as equating two ways round a cube or a hexagonal prism, then this verification is a matter of taking a suitable collection of such cubes and prisms for β and α and sticking them together in the right way. When realised in two dimensions, this amounts to displaying a succession of equalities of rather large pasting diagrams. We leave the task of reconstructing these to the reader.

Let us consider now the middle-four interchange axiom. Asking for this be satisfied amounts to checking that the other obvious way of defining $\beta * \alpha$ – via GF' rather than $G'F$ – gives the same answer; and this follows quickly from the middle-four interchange law in the hom-bicategories of \mathcal{U} , and the first icon axiom for β .

It remains to give the associativity and unit constraints a , l and r for **Tricat**₂. For the left unit constraint l , consider a lax homomorphism $F: \mathcal{S} \rightarrow \mathcal{T}$, and write F' for the composite $\text{id}_{\mathcal{T}}.F: \mathcal{S} \rightarrow \mathcal{T}$. Now, F' agrees with F on 0-cells and on

hom-bicategories, but differs in the remaining coherence data; indeed, we have

$$\iota_A^{F'} = I_{FA} \xRightarrow{\text{id}_{I_{FA}}} I_{FA} \xRightarrow{\iota_A^F} FI_A$$

$$\text{and } \chi_{f,g}^{F'} = Fg.Ff \xRightarrow{\text{id}_{Fg.Ff}} Fg.Ff \xRightarrow{\chi_{f,g}^F} F(gf).$$

Thus we define a 2-cell $l_F: \text{id}_{\mathcal{T}}.F \Rightarrow F$ in **Tricat**₂ as follows:

- (TD1) $(l_F)_{A,B} = \text{id}_{F_{A,B}}: F_{A,B} \Rightarrow F_{A,B}$;
- (TD2) $M_A^{l_F}$ is the unit isomorphism $\iota_A^F.(\text{id}_{I_{FA}}) \Rightarrow \iota_A^F$ in the bicategory $\mathcal{T}(FA, FA)$;
- (TD3) $\Pi_{f,g}^{l_F}$ is the unit isomorphism $\chi_{f,g}^F.(\text{id}_{Fg.Ff}) \Rightarrow \chi_{f,g}^F$ in the bicategory $\mathcal{T}(FA, FC)$.

Now each axiom (TA1)–(TA4) is a tautology which describes how we obtained $\chi^{F'}$, $\delta^{F'}$, $\gamma^{F'}$ and $\omega^{F'}$ from the corresponding data for F . The definition of r is dual to that of l , so we pass over it and onto the associativity constraint a . Consider three lax homomorphisms $F: \mathcal{R} \rightarrow \mathcal{S}$, $G: \mathcal{S} \rightarrow \mathcal{T}$ and $H: \mathcal{T} \rightarrow \mathcal{U}$ and the two composites $(HG)F$ and $H(GF): \mathcal{R} \rightarrow \mathcal{U}$. As above, these agree on 0-cells and on hom-bicategories (and so we write their common value simply as HGF) but differ with respect to coherence data. This time we have:

$$\begin{aligned} \iota^{(HG)F} &= HG\iota^F.(H\iota^G.\iota^H), & \iota^{H(GF)} &= (HG\iota^F.H\iota^G).\iota^H, \\ \chi^{(HG)F} &= HG\chi^F.(H\chi^G.\chi^H) & \text{and} & \quad \chi^{H(GF)} = (HG\chi^F.H\chi^G).\chi^H, \end{aligned}$$

where we omit the subscripts for clarity. Thus we take $a_{F,G,H}: (HG)F \Rightarrow H(GF)$ in **Tricat**₂ to be:

- (TD1) $(a_{F,G,H})_{A,B} = \text{id}_{(HGF)_{A,B}}: (HGF)_{A,B} \Rightarrow (HGF)_{A,B}$;
- (TD2) $M_A^{a_{F,G,H}}$ is the associativity isomorphism

$$HG\iota_A^F.(H\iota_{FA}^G.\iota_{GFA}^H) \Rightarrow (HG\iota_A^F.H\iota_{FA}^G).\iota_{GFA}^H$$

in the bicategory $\mathcal{U}(HGFA, HGFA)$;

- (TD3) $\Pi_{f,g}^{a_{F,G,H}}$ is the associativity isomorphism

$$HG\chi_{f,g}^F.(H\chi_{Ff,Fg}^G.\chi_{GFf,GFg}^H) \Rightarrow (HG\chi_{f,g}^F.H\chi_{Ff,Fg}^G).\chi_{GFf,GFg}^H$$

in the bicategory $\mathcal{U}(HGFA, HGFC)$.

We must now verify axioms (TA1)–(TA4) for these data. Observe first that the 3-cell data χ , γ , δ and ω for $H(GF)$ and for $(HG)F$ are, in fact, obtained as different bracketings of the same pasting diagram; for example, both $\omega_{f,g,h}^{H(GF)}$ and $\omega_{f,g,h}^{(HG)F}$ are obtained as pastings of the diagram on the following page. So by the pasting theorem for bicategories, we can obtain the 3-cell data χ , γ , δ and ω for $H(GF)$ from that for $(HG)F$ by pasting with suitable associativity isomorphisms in the

appropriate hom-bicategory of \mathcal{U} ; and this is precisely what axioms (TA1)–(TA4) say.

It remains to check the naturality of l , r and a , and the pentagon and triangle identities. For the naturality of l , we must show that for any 2-cell $\alpha: F \Rightarrow G$ of **Tricat**₂, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{id}_{\mathcal{T}}.F & \xRightarrow{l_F} & F \\ \mathrm{id}_{\mathcal{T}}.\alpha \Downarrow & & \Downarrow \alpha \\ \mathrm{id}_{\mathcal{T}}.G & \xRightarrow{l_G} & G. \end{array}$$

We easily verify that the left-hand 2-cell $\alpha' = \mathrm{id}_{\mathcal{T}}.\alpha$ has components $\alpha'_\theta = \alpha_\theta$, $M_A^{\alpha'} = M_A^\alpha \cdot (\mathrm{id}_{I_{FA}})$ and $\Pi_{f,g}^{\alpha'} = \Pi_{f,g}^\alpha \cdot (\mathrm{id}_{Ff.Fg})$; therefore the naturality of l is a consequence of the naturality of the left unit constraints in the hom-bicategories of \mathcal{T} ; and dually for r . For the naturality of a , we must show that the following diagram commutes in **Tricat**₂ for all suitable 2-cells α , β and ϵ :

$$\begin{array}{ccc} (HG)F & \xRightarrow{a_{F,G,H}} & H(GF) \\ (\alpha\beta)\epsilon \Downarrow & & \Downarrow \alpha(\beta\epsilon) \\ (H'G')F' & \xRightarrow{a_{F',G',H'}} & H'(G'F'), \end{array}$$

for which we must show that (TD1)–(TD3) agree for the two ways around this square. For (TD1) this is trivial; so consider (TD2). For both $(\alpha\beta)\epsilon$ and $\alpha(\beta\epsilon)$, we obtain this datum by pasting together the same 3×3 diagram of 3-cells; the only difference being the manner in which we bracket together the boundary of this diagram. Thus the commutativity of the above square with respect to (TD2) is a further instance of the pasting theorem for bicategories. (TD3) is obtained in a similar manner.

Finally, it is not hard to verify that the pentagon and triangle identities for a , l and r follow from instances of the pentagon and triangle identities in the hom-bicategories of the target tricategory. This completes the definition of the bicategory **Tricat**₂.

$$\begin{array}{ccccccc}
& & HGF(hg).HGFf \xRightarrow{\chi^H} H(GF(hg).GFf) \xRightarrow{H\chi^G} HG(F(hg).Ff) \xRightarrow{HG\chi^F} HGF((hg)f) \\
& \nearrow^{HG\chi^F.1} & \Downarrow_{(\chi^H)^{-1}} & \nearrow^{H(G\chi^F.1)} & \Downarrow_{H(\chi^G)^{-1}} & \nearrow^{HG(\chi^F.1)} & \\
& & HG(Fh.Fg).HGFf \xRightarrow{\chi^H} H(G(Fh.Fg).GFf) \xRightarrow{H\chi^G} HG((Fh.Fg).Ff) & & \Downarrow_{HG\omega^F} & & HGF(h(gf)) \\
& \nearrow^{H\chi^G.1} & \Downarrow_{(\chi^H)^{-1}} & \nearrow^{H(\chi^G.1)} & \searrow^{HG\alpha} & & \\
H(GFh.GFg).HGFf \xRightarrow{\chi^H} H((GFh.GFg).GFf) & & \Downarrow_{H\omega^G} & & HG(Fh.(Fg.Ff)) \xRightarrow{HG(1.\chi^F)} HG(Fh.F(gf)) & & \\
\swarrow^{\chi^H.1} & \searrow^{H\alpha} & & \nearrow^{H\chi^G} & \Downarrow_{H\chi^G} & \nearrow^{H\chi^G} & \\
(HGFh.HGFg).HGFf & & \Downarrow_{\omega^H} & & H(GFh.G(Fg.Ff)) \xRightarrow{H(1.\chi^G)} H(GFh.G(Fg.Ff)) \xRightarrow{H(1.G\chi^F)} H(GFh.GF(gf)) & & \\
& \searrow^{\alpha} & & \nearrow^{\chi^H} & \Downarrow_{\chi^H} & \nearrow^{\chi^H} & \Downarrow_{\chi^H} & \nearrow^{\chi^H} \\
HGFh.(HGFg.HGFf) \xRightarrow{1.\chi^H} HGFh.H(GFg.GFf) \xRightarrow{1.H\chi^G} HGFh.HG(Fg.Ff) \xRightarrow{1.HG\chi^F} HGFh.HGF(gf) & & & & & &
\end{array}$$

3 Locally double bicategories

We would now like to take the bicategory of tricategories \mathbf{Tricat}_2 and extend it to a tricategory of tricategories \mathbf{Tricat}_3 . This will have the same 0-cells and 1-cells as \mathbf{Tricat}_2 , but will have 2-cells with one less level of degeneracy, which consequently admit a notion of 3-cell between them.

However, there is a problem with this. From the perspective of \mathbf{Tricat}_2 , the composition of lax homomorphisms is associative up-to-isomorphism; but when we move to \mathbf{Tricat}_3 , this same composition becomes associative only up-to-*equivalence*, even though the constraint 2-cells are the same in both cases. The reason is that 2-cell composition in \mathbf{Tricat}_3 is less strict than it is in \mathbf{Tricat}_2 , and so the isomorphisms which witness associativity in \mathbf{Tricat}_2 become mere equivalences in \mathbf{Tricat}_3 .

To resolve this problem, we will describe a richer structure than a tricategory into which tricategories organize themselves. This structure also has 0-, 1-, 2- and 3-cells, but the 2-cells now come in two varieties, allowing us to capture both the doubly-degenerate 2-cells of \mathbf{Tricat}_2 with their strictly associative composition and the singly-degenerate ones of \mathbf{Tricat}_3 with their up-to-isomorphism composition. Moreover, the associativity and unit constraints of this structure are given by 2-cells of the first, rather than the second, type, and thus are of a bicategorical, up-to-isomorphism, rather than a tricategorical, up-to-equivalence, kind.

What we have, then, is not a tricategory of tricategories, but a *locally double bicategory* of them. A locally double bicategory may be described very succinctly as a “bicategory weakly enriched in weak double categories”; and in order to expand upon this we must first recall some relevant concepts pertaining to weak double categories.

The concept of *strict* double category is due to Ehresmann. It is an example of the notion of *double model* for an essentially-algebraic theory, this being a model of the theory in its own category of (**Set**-based) models. Thus a double category – which is a double model of the theory of categories – is a category object in **Cat**.

Now, the theory of categories is somewhat special, since its category of (**Set**-based) models may be enriched to a 2-category, so that, as well as *strict* category objects in **Cat**, we may also consider *pseudo* category objects: and these are the weak double categories which we will be interested in.

Definition 2. A **weak double category** \mathfrak{C} is given by specifying a collection of *objects* x, y, z, \dots , a collection of *vertical 1-cells* between objects, which we write as $a: x \rightarrow y$, a collection of *horizontal 1-cells* between objects, which we write as $f: x \rightarrowtail y$, and a collection of *2-cells*, each of which is bounded by a square of horizontal and vertical arrows, and which we write as:

$$\begin{array}{ccc} x & \xrightarrow{f} & w \\ a \downarrow & \Downarrow \alpha & \downarrow b \\ y & \xrightarrow{g} & z, \end{array}$$

or sometimes simply as $\alpha: f \Rightarrow g$. Moreover, we must give:

- Identities and composition for vertical 1-cells, $\text{id}_x: x \rightarrow x$ and $(a, b) \mapsto ab$, making the objects and vertical arrows into a category \mathcal{C}_0 ;
- Vertical identities and composition for 2-cells, $\text{id}_f: f \Rightarrow f$ and $(\beta, \alpha) \mapsto \beta\alpha$:

$$\begin{array}{ccc}
\begin{array}{ccc} x & \xrightarrow{f} & y \\ \text{id}_x \downarrow & \Downarrow \text{id}_f & \downarrow \text{id}_y \\ x & \xrightarrow{f} & y \end{array} & ; & \begin{array}{ccc} u & \xrightarrow{f} & x \\ a \downarrow & \Downarrow \alpha & \downarrow b \\ v & \xrightarrow{g} & y \\ c \downarrow & \Downarrow \beta & \downarrow d \\ w & \xrightarrow{h} & z \end{array} \mapsto \begin{array}{ccc} u & \xrightarrow{f} & x \\ ca \downarrow & \Downarrow \beta\alpha & \downarrow db \\ w & \xrightarrow{h} & z \end{array}
\end{array}$$

making the horizontal arrows and 2-cells into a category \mathcal{C}_1 for which “vertical source” and “vertical target” become functors $s, t: \mathcal{C}_1 \rightarrow \mathcal{C}_0$;

- Identities and composition for horizontal 1-cells, $I_x: x \rightarrow x$ and $(g, f) \mapsto gf$;
- Horizontal identities and composition for 2-cells, $I_a: I_x \Rightarrow I_y$ and $(\beta, \alpha) \mapsto \beta * \alpha$:

$$\begin{array}{ccc}
\begin{array}{ccc} x & \xrightarrow{I_x} & x \\ a \downarrow & \Downarrow I_a & \downarrow a \\ y & \xrightarrow{I_y} & y \end{array} & ; & \begin{array}{ccc} u & \xrightarrow{f} & v \xrightarrow{g} w \\ a \downarrow & \Downarrow \alpha & \downarrow b \Downarrow \beta \downarrow c \\ x & \xrightarrow{h} & y \xrightarrow{k} z \end{array} \mapsto \begin{array}{ccc} u & \xrightarrow{gf} & w \\ a \downarrow & \Downarrow \beta * \alpha & \downarrow c \\ x & \xrightarrow{kh} & z \end{array},
\end{array}$$

satisfying functoriality constraints: firstly, $I_{(-)}$ is a functor $\mathcal{C}_0 \rightarrow \mathcal{C}_1$, which says that $I_{\text{id}_x} = \text{id}_{I_x}$ and $I_{ab} = I_a \cdot I_b$ and secondly, horizontal composition is a functor $*$: $\mathcal{C}_1 \times_s \times_t \mathcal{C}_1 \rightarrow \mathcal{C}_1$ which says that $\text{id}_g * \text{id}_f = \text{id}_{gf}$ and $(\delta * \gamma) \cdot (\beta * \alpha) = (\delta\beta) * (\gamma\alpha)$.

- Horizontal unitality and associativity constraints given by 2-cells

$$\begin{array}{ccc}
\begin{array}{ccc} x & \xrightarrow{I_y \cdot f} & y \\ \text{id}_x \downarrow & \Downarrow \text{id}_f & \downarrow \text{id}_y \\ x & \xrightarrow{f} & y \end{array} & , & \begin{array}{ccc} x & \xrightarrow{f \cdot I_x} & y \\ \text{id}_x \downarrow & \Downarrow \text{id}_f & \downarrow \text{id}_y \\ x & \xrightarrow{f} & y \end{array} , \quad \text{and} \quad \begin{array}{ccc} x & \xrightarrow{h(gf)} & z \\ \text{id}_x \downarrow & \Downarrow \alpha_{f,g,h} & \downarrow \text{id}_z \\ x & \xrightarrow{(hg)f} & z \end{array},
\end{array}$$

natural in f, g and h , and invertible as arrows of \mathcal{C}_1 . These 2-cells must obey two laws: the pentagon law, which equates the two routes from $k(h(gf))$ to $((kh)g)f$, and the triangle law, which equates the two routes from $g \cdot (I_y \cdot f)$ to gf .

A more comprehensive reference on weak double categories and related matters is [9]: though be aware that we interchange its usage of the terms “horizontal” and “vertical” to give a better fit with the usual 2-categorical terminology. Some simple examples of weak double categories are \mathbf{Cat} , the weak double category of “categories, functors, profunctors and transformations”, \mathbf{Rng} , the weak double category of “rings, ring homomorphisms, bimodules and skew-linear maps”, and the weak double category $\mathbf{Span}(\mathcal{C})$ of “objects, morphisms, spans and span morphisms” in a

category with pullbacks \mathcal{C} . These are typical examples of weak double categories, in that they have notions of *homomorphism* and *bimodule* as their respective vertical and horizontal 1-cells. Any bicategory \mathcal{B} gives us a weak double category $\mathbb{U}(\mathcal{B})$ with only identity vertical 1-cells, whilst any weak double category \mathfrak{C} gives a bicategory $\mathbb{H}(\mathfrak{C})$ upon throwing away the non-identity vertical 1-cells, and all the 2-cells except for the **globular 2-cells**, whose vertical source and target are identity arrows.

Just as in the theory of bicategories, the appropriate notion of morphism between weak double categories only preserves horizontal composition up to comparison 2-cells, the most important case being the *homomorphisms*, for which these 2-cells are invertible. We can define a homomorphism between small weak double categories in terms of a pseudomorphism of pseudocategory objects, but just as easy is to give the elementary description:

Definition 3. A **homomorphism of weak double categories** $F: \mathfrak{C} \rightarrow \mathfrak{D}$ is given by assignments on objects, 1-cells and 2-cells which preserve source and target and are functorial with respect to vertical composition of 1- and 2-cells, together with comparison 2-cells

$$\begin{array}{ccc} Fx & \xrightarrow{I_{Fx}} & Fx \\ \text{id}_{Fx} \downarrow & \Downarrow \mathfrak{m}_x & \downarrow \text{id}_{Fx} \\ Fx & \xrightarrow{FI_x} & Fx \end{array} \quad \text{and} \quad \begin{array}{ccc} Fx & \xrightarrow{Fg.Ff} & Fz \\ \text{id}_{Fx} \downarrow & \Downarrow \mathfrak{m}_{f,g} & \downarrow \text{id}_{Fz} \\ Fx & \xrightarrow{F(gf)} & Fz \end{array}$$

which are invertible as arrows of \mathcal{D}_1 , and natural in x , respectively g and f . Moreover, we require the commutativity of three familiar diagrams, which equate, respectively, the two possible ways of going from $Ff.I_{Fx}$ to Ff , from $I_{Fy}.Ff$ to Ff , and from $Fh.(Fg.Ff)$ to $F((hg)f)$.

With the obvious notion of composition and identities, we obtain a category **DblCat** of (possibly large) weak double categories and homomorphisms between them. If we write **Bicat** for the category of bicategories and homomorphisms, then the assignments $\mathcal{B} \mapsto \mathbb{U}(\mathcal{B})$ and $\mathfrak{C} \mapsto \mathbb{H}(\mathfrak{C})$ described above extend to a pair of adjoint functors $\mathbb{U} \dashv \mathbb{H}: \mathbf{DblCat} \rightarrow \mathbf{Bicat}$, for which the composite $\mathbb{H}\mathbb{U}$ is the identity; we can thus view **Bicat** as a coreflective subcategory of **DblCat**.

Now, **DblCat** is in fact the underlying ordinary category of a 2-category whose 2-cells are the so-called *vertical transformations*. We can understand these 2-cells by observing that there is a 2-monad (or even a **Cat**-operad) on the 2-category **CatGph** := $[\bullet \rightrightarrows \bullet, \mathbf{Cat}]$ whose strict algebras are small weak double categories, and whose algebra pseudomorphisms are the homomorphisms between them; now the corresponding *algebra 2-cells* are precisely the vertical transformations. Spelling this out, we have:

Definition 4. A **vertical transformation** $\alpha: F \Rightarrow G$ between homomorphisms of weak double categories $F, G: \mathfrak{C} \rightarrow \mathfrak{D}$ is given by specifying, for each object $x \in \mathfrak{C}$, a vertical 1-cell $\alpha_x: Fx \rightarrow Gx$ of \mathfrak{D} and for each horizontal 1-cell $f: x \rightarrowtail y$ in \mathfrak{C} a

2-cell

$$\begin{array}{ccc} Fx & \xrightarrow{Ff} & Fy \\ \alpha_x \downarrow & \Downarrow \alpha_f & \downarrow \alpha_y \\ Gx & \xrightarrow{Gf} & Gy \end{array}$$

of \mathfrak{D} , such that the α_x 's are natural in morphisms of \mathcal{D}_0 , the α_f 's are natural in morphisms of \mathcal{D}_1 , and the following diagrams commute:

$$\begin{array}{ccc} I_{Fx} & \xrightarrow{m_x^F} & FI_x \\ I_{\alpha_x} \Downarrow & & \Downarrow \alpha_{I_x} \\ I_{Gx} & \xrightarrow{m_x^G} & GI_x \end{array} \quad \text{and} \quad \begin{array}{ccc} Fg.Ff & \xrightarrow{m_{g,f}^F} & F(gf) \\ \alpha_g * \alpha_f \Downarrow & & \Downarrow \alpha_{gf} \\ Gg.Gf & \xrightarrow{m_{g,f}^G} & G(gf). \end{array}$$

If we restrict our attention to the bicategories inside **DbICat**, then the vertical transformation between homomorphisms are precisely the *icons* of section 1; however, the reader should carefully note that the coreflection of **DbICat** into **Bicat** does *not* enrich to a two-dimensional coreflection, since there is no way of coreflecting a general vertical transformation between homomorphisms of weak double categories into an icon between the corresponding homomorphisms of bicategories. The question of what we can say about this situation is one we return to in Section 5.

It follows from the algebraic description of **DbICat** that it admits a wide class of 2-dimensional limits, of which we will only be concerned with finite products. That **DbICat** admits these, makes it, of course, into a symmetric monoidal category, but the 2-dimensional aspect of these products give us a *symmetric monoidal 2-category*: that is, a symmetric monoidal category whose tensor product is a 2-functor and whose coherence natural transformations are 2-natural transformations. What we now wish to describe is how we can use this monoidal 2-category **DbICat** as a suitable base for *enrichment*.

For any monoidal category \mathcal{V} , we have the well-known notion of a *category enriched in \mathcal{V}* , which instead of having hom-sets between 0-cells, has hom-objects drawn from \mathcal{V} , with the corresponding composition being expressed by morphisms of \mathcal{V} obeying laws assuring associativity and unitality. Slightly less well-known is the two-dimensional generalisation of this notion: if instead of a monoidal category \mathcal{V} , we begin with a *monoidal bicategory* \mathcal{W} in the sense of [5], we obtain a notion of *bicategory enriched in \mathcal{W}* or *\mathcal{W} -bicategory*⁴, which instead of hom-categories between 0-cells has hom-objects drawn from \mathcal{W} ; the corresponding composition is once again expressed by morphisms of \mathcal{W} , which are now required to be associative and unital only up to coherent 2-cells.

Just as a (locally small) category can be viewed as a category enriched in **Set**, so a (locally small) bicategory can be viewed as a bicategory enriched in **Cat**. Other

⁴Warning! This is emphatically *not* the same as the notion of “category enriched in a bicategory” studied in [18]: this latter is really just an example of a category enriched in a monoidal category where the monoidal category happens to be spread out over many objects.

straightforward examples are obtained by taking $\mathcal{W} = \mathcal{V}\text{-}\mathbf{Cat}$ for some monoidal category \mathcal{V} , for which a \mathcal{W} -bicategory has sets of 0- and 1- cells as usual, but now a \mathcal{V} -object of 2-cells between any parallel pair of 1-cells; by taking $\mathcal{W} = \mathbf{Mod}$, the bicategory of categories and profunctors, for which a \mathcal{W} -bicategory is a *probi-category* in the sense of Day [4]; and by taking \mathcal{W} to be an ordinary monoidal category, viewed as a locally discrete monoidal bicategory, whereupon \mathcal{W} -bicategories reduce to categories enriched in \mathcal{W} . An account of the general theory of enriched bicategories can be found in [14], but we will need sufficiently little of it that our account can be considered to be self-contained. Indeed, we will only consider in detail the case where \mathcal{W} is the monoidal 2-category \mathbf{DbICat} described above.

Definition 5. A **locally double bicategory** or **DbICat-bicategory** \mathfrak{B} is given by the following data:

(LDD1) A collection $\text{ob } \mathfrak{B}$ of objects;

(LDD2) For every pair $A, B \in \text{ob } \mathfrak{B}$, a weak double category $\mathfrak{B}(A, B)$;

(LDD3) For every $A \in \text{ob } \mathfrak{B}$, a unit homomorphism

$$\lrcorner I_A \lrcorner : 1 \rightarrow \mathfrak{B}(A, A);$$

(LDD4) For every triple $A, B, C \in \text{ob } \mathfrak{B}$, a composition homomorphism

$$\otimes : \mathfrak{B}(B, C) \times \mathfrak{B}(A, B) \rightarrow \mathfrak{B}(A, C);$$

(LDD5) For every pair $A, B \in \text{ob } \mathfrak{B}$, invertible vertical transformations

$$\begin{array}{ccccc} \mathfrak{B}(A, B) \times \mathfrak{B}(A, A) & \xleftarrow{1 \times \lrcorner I_A \lrcorner} & \mathfrak{B}(A, B) & \xrightarrow{\lrcorner I_B \lrcorner \times 1} & \mathfrak{B}(B, B) \times \mathfrak{B}(A, B); \\ & \searrow \scriptstyle \otimes & \downarrow \scriptstyle 1 & \swarrow \scriptstyle \otimes & \\ & & \mathfrak{B}(A, B) & & \end{array}$$

(LDD6) For every quadruple $A, B, C, D \in \text{ob } \mathfrak{B}$, an invertible vertical transformation

$$\begin{array}{ccc} \mathfrak{B}(C, D) \times \mathfrak{B}(B, C) \times \mathfrak{B}(A, B) & \xrightarrow{1 \times \otimes} & \mathfrak{B}(C, D) \times \mathfrak{B}(A, C) \\ \otimes \times 1 \downarrow & \Downarrow a & \downarrow \otimes \\ \mathfrak{B}(B, D) \times \mathfrak{B}(A, B) & \xrightarrow{\otimes} & \mathfrak{B}(A, D). \end{array}$$

This data is subject to the following two axioms:

(LDA1) For each triple of objects A, B, C of \mathfrak{B} , the following pasting equality holds:

$$\begin{array}{ccccc} \mathfrak{B}^2 & \xrightarrow{1 \times \lrcorner I_B \lrcorner \times 1} & \mathfrak{B}^3 & \xrightarrow{1 \times \otimes} & \mathfrak{B}^2 \\ \downarrow \scriptstyle 1 & \Downarrow \scriptstyle 1 \times l & \downarrow \scriptstyle 1 \times \otimes & & \downarrow \scriptstyle \otimes \\ \mathfrak{B}^2 & \xrightarrow{1} & \mathfrak{B}^2 & \xrightarrow{\otimes} & \mathfrak{B} \end{array} = \begin{array}{ccccc} \mathfrak{B}^2 & \xrightarrow{1 \times \lrcorner I_B \lrcorner \times 1} & \mathfrak{B}^3 & \xrightarrow{1 \times \otimes} & \mathfrak{B}^2 \\ \downarrow \scriptstyle 1 & \Downarrow \scriptstyle r \times 1 & \downarrow \scriptstyle \otimes \times 1 & \Downarrow \scriptstyle a & \downarrow \scriptstyle \otimes \\ \mathfrak{B}^2 & \xrightarrow{1} & \mathfrak{B}^2 & \xrightarrow{\otimes} & \mathfrak{B}, \end{array}$$

where \mathfrak{B}^2 and \mathfrak{B}^3 abbreviate the appropriate products of hom-double categories;

(LDA2) For each quintuple of objects A, B, C, D, E of \mathfrak{B} , the following pasting equality holds:

$$\begin{array}{ccc}
 \mathfrak{B}^4 & \xrightarrow{1 \times 1 \times \otimes} & \mathfrak{B}^3 \\
 \downarrow \otimes \times 1 \times 1 & \cong & \downarrow \otimes \times 1 \\
 \mathfrak{B}^3 & \xrightarrow{1 \times \otimes} & \mathfrak{B}^2 \\
 \downarrow \otimes \times 1 & \searrow a & \downarrow \otimes \\
 \mathfrak{B}^2 & \xrightarrow{\otimes} & \mathfrak{B}
 \end{array}
 =
 \begin{array}{ccc}
 \mathfrak{B}^4 & \xrightarrow{1 \times 1 \times \otimes} & \mathfrak{B}^3 \\
 \downarrow \otimes \times 1 \times 1 & \searrow 1 \times \otimes \times 1 & \downarrow 1 \times \otimes \\
 \mathfrak{B}^3 & \xrightarrow{a \times 1} & \mathfrak{B}^3 \\
 \downarrow \otimes \times 1 & \searrow \otimes \times 1 & \downarrow \otimes \\
 \mathfrak{B}^2 & \xrightarrow{\otimes} & \mathfrak{B}
 \end{array}$$

where we observe the same convention regarding \mathfrak{B}^4 , \mathfrak{B}^3 and \mathfrak{B}^2 .

It may be helpful to extract a description of the various sorts of composition that a **DblCat**-bicategory possesses. The 0-cells, 1-cells and vertical 2-cells form an ordinary bicategory. Next come the horizontal 2-cells, which can be composed with each other along either a 1-cell boundary or a 0-cell boundary, with both compositions being associative up to an invertible 3-cell; moreover, the corresponding “middle four interchange” law only holds up to an invertible 3-cell. Finally, the 3-cells themselves can be composed with each other along the two different types of 2-cell boundary and along 0-cell boundaries; and these operations are strictly associative modulo the associativity of the boundaries.

A *one-object* locally double bicategory amounts to a **monoidal double category** [7, 10, 17] – that is, a weak double category with an up-to-isomorphism tensor product on it. In particular, any double category with *finite products* in the appropriate double categorical sense⁵ becomes a monoidal double category under the cartesian tensor. The double categories **Cat**, **Span**(\mathcal{C}) (where \mathcal{C} is a category with finite limits) and **Ang** are all monoidal in this way: though in the case of **Ang**, one would more usually be interested in the other natural monoidal structure which is derived from the usual tensor product on the category of rings.

For a non-degenerate example of a locally double bicategory, we turn to **DblCat** itself. As demonstrated in [7], we may define an internal hom 2-functor

$$\mathfrak{Hom}(-, ?): \mathbf{DblCat}^{\text{op}} \times \mathbf{DblCat} \rightarrow \mathbf{DblCat}$$

for which $\mathfrak{Hom}(\mathfrak{C}, \mathfrak{D})$ is the following double category. Its objects are homomorphisms $\mathfrak{C} \rightarrow \mathfrak{D}$, and its vertical 1-cells $\alpha: F \Rightarrow G$ are the vertical transformations between them. Its horizontal 1-cells $\alpha: F \Rrightarrow G$ are the *horizontal pseudo-natural transformations*, whose components at an object $x \in \mathfrak{C}$ are given by horizontal

⁵By which we mean a *pseudo-functorial choice of double products* in the sense of [9]. Such weak double categories are slightly stricter versions of the *cartesian bicategories* of [2].

1-cells $\alpha_x: Fx \rightarrow Gx$ of \mathfrak{D} , satisfying naturality conditions like those for a pseudo-natural transformation between homomorphisms of bicategories; and indeed, in the case that \mathfrak{C} and \mathfrak{D} are themselves bicategories the two notions coincide. Finally, the 2-cells of $\mathfrak{Hom}(\mathfrak{C}, \mathfrak{D})$ are the *cubical modifications*, which are bounded by two horizontal and two vertical transformations and whose basic data consists of giving, for each object of the source, a 2-cell of the target bounded by the components of these transformations.

When we say that \mathfrak{Hom} acts as an internal hom, we are affirming a universal property: namely, that for each \mathfrak{C} the 2-functor $(-) \times \mathfrak{C}: \mathbf{DbICat} \rightarrow \mathbf{DbICat}$ is left biadjoint to $\mathfrak{Hom}(\mathfrak{C}, -)$, so that what we have is a *biclosed* monoidal bicategory in the sense of [5].

Now, in [14], it is demonstrated that, just as any closed monoidal category can be viewed as a category enriched over itself, so can any biclosed monoidal bicategory be viewed as a bicategory enriched over itself, with the hom-objects being given by the biclosed structure. Applying this result to the 2-category \mathbf{DbICat} , we obtain a locally double bicategory \mathbf{DbICat} , with 0-cells being the weak double categories; 1-cells, the homomorphisms; vertical 2-cells, the vertical transformations; horizontal 2-cells, the horizontal pseudo-natural transformations; and 3-cells the cubical modifications.

Moreover, we may restrict our attention to the *bicategories* inside \mathbf{DbICat} to obtain:

Corollary 6. *There is a locally double bicategory \mathbf{Bicat} which has as 0-cells, bicategories; as 1-cells, homomorphisms; as vertical 2-cells, icons; as horizontal 2-cells, pseudo-natural transformations; and as 3-cells, cubical modifications.*

These particular cubical modifications are the same ones that we mentioned in Section 2, and we record their definition for future use:

Definition 7. Let $F, G, H, K: \mathcal{B} \rightarrow \mathcal{C}$ be homomorphisms of bicategories; let $\alpha: F \rightrightarrows G$ and $\beta: H \rightrightarrows K$ be pseudo-natural transformations; and let $\gamma: F \Rightarrow H$ and $\delta: G \Rightarrow K$ be icons. Then a **cubical modification**

$$\begin{array}{ccc} F & \xrightleftharpoons{\alpha} & G \\ \gamma \Downarrow & \Downarrow \Gamma & \Downarrow \delta \\ H & \xrightleftharpoons[\beta]{} & K \end{array}$$

is given by specifying, for every object $A \in \mathcal{B}$, a 2-cell $\Gamma_A: \alpha_A \Rightarrow \beta_A$, such that for

every 1-cell $f: A \rightarrow B$ of \mathcal{B} , the following pasting equality holds:

$$\begin{array}{ccc}
 & FB \xrightarrow{\alpha_B} GB & \\
 Ff \nearrow & \Downarrow \alpha_f & \nwarrow Gf \\
 FA \xrightarrow{\alpha_A} GA & & KB \\
 \Downarrow \Gamma_A & \Downarrow \delta_f & \\
 HA \xrightarrow{\beta_A} KA & & KB \\
 & Kf \nearrow &
 \end{array}
 =
 \begin{array}{ccc}
 & FB \xrightarrow{\alpha_B} GB & \\
 Ff \nearrow & \Downarrow \Gamma_B & \nwarrow \\
 FA \xrightarrow{\gamma_f} HB & & KB \\
 \Downarrow \gamma_f & \Downarrow \beta_f & \\
 HA \xrightarrow{\beta_A} KA & & KB \\
 & Kf \nearrow &
 \end{array}$$

If we set γ and δ to be identity icons in the above definition, we recapture the standard notion of a modification between pseudo-natural transformations, and thus the locally double bicategory \mathbf{Bicat} is rich enough to encode all of the structure of the tricategory of bicategories, but is able to do so using coherence whose complexity does not rise above the bicategorical level.

Pleasing as this is, we should note that not every tricategory can be reduced to a locally double bicategory in this way; for example, given a bicategory \mathcal{B} with bipullbacks, we may form the tricategory $\mathbf{Span}(\mathcal{B})$ of spans in \mathcal{B} . In this tricategory, 1-cell composition is given by bipullback, and so is only determined up-to-equivalence, rather than up-to-isomorphism; so evidently, it will be inexpressible as a locally double bicategory.

Remark 8. There are two canonical ways of forming a tricategory of bicategories, corresponding to the two canonical ways of composing a pair of strong transformations along a 0-cell boundary: however, Proposition 6 exhibited a single canonical locally double bicategory of bicategories. The discrepancy is resolved if we observe that to obtain this \mathbf{DbICat} -bicategory we must fix a choice of biclosed structure on \mathbf{DbICat} , and that there are two canonical ways of doing this, depending on how we choose the counit maps $\mathfrak{Hom}(\mathfrak{B}, \mathfrak{C}) \times \mathfrak{B} \rightarrow \mathfrak{C}$ for the biadjunctions in question.

4 A locally double bicategory of tricategories

We now return to our study of tricategories with the goal of forming them into a locally double bicategory \mathfrak{Tricat}_3 . The 0-, 1- and 2-cells of the bicategory \mathbf{Tricat}_2 will provide us with the 0- and 1-cells and the vertical 2-cells of \mathfrak{Tricat}_3 , so it remains only to introduce the horizontal 2-cells and the 3-cells, which we will call *pseudo-icons* and *pseudo-icon modifications* respectively. In fact, following our philosophy of being as lax as possible, we begin by considering the more general *oplax icons*:

Definition 9. Let there be given lax homomorphisms of tricategories $F, G: \mathcal{S} \rightarrow \mathcal{T}$; then an **oplax icon** $\alpha: F \Rightarrow G$ may exist only if F and G agree on objects whereupon it consists of the following data:

(ID1) For each A and B in \mathcal{S} , an oplax natural transformation

$$\alpha_{A,B}: F_{A,B} \Rightarrow G_{A,B}: \mathcal{S}(A, B) \rightarrow \mathcal{T}(FA, FB)$$

(and in particular, for each 1-cell $f: A \rightarrow B$ of \mathcal{S} , we have a 2-cell $\alpha_f: Ff \Rightarrow Gf$ of \mathcal{T} , and for each 2-cell $\theta: f \Rightarrow g$ of \mathcal{S} , a 3-cell

$$\begin{array}{ccc} Ff & \xRightarrow{F\theta} & Fg \\ \alpha_f \Downarrow & \Downarrow \alpha_\theta & \Downarrow \alpha_g \\ Gf & \xRightarrow{G\theta} & Gg \end{array});$$

(ID2) For each object A of \mathcal{S} , a 3-cell of \mathcal{T} :

$$\begin{array}{ccc} I_{FA} & \xRightarrow{\iota_A^F} & FI_A \\ \parallel & \Downarrow M_A^\alpha & \Downarrow \alpha_{I_A} \\ I_{GA} & \xRightarrow{\iota_A^G} & GI_A; \end{array}$$

(ID3) For each A, B and C in \mathcal{S} , a modification

$$\begin{array}{ccc} F(-) \otimes F(?) & \xRightarrow{\chi^F} & F((-) \otimes (?)) \\ \alpha_{(-) \otimes (?)} \Downarrow & \Downarrow \Pi_{A,B,C}^\alpha & \Downarrow \alpha_{((-) \otimes (?))} \\ G(-) \otimes G(?) & \xRightarrow{\chi^G} & G((-) \otimes (?)), \end{array}$$

where, for instance, $F(-) \otimes F(?)$ represents the homomorphism

$$\mathcal{S}(B, C) \times \mathcal{S}(A, B) \xrightarrow{F \times F} \mathcal{T}(FB, FC) \times \mathcal{T}(FA, FB) \xrightarrow{\otimes} \mathcal{T}(FA, FC)$$

(and in particular, for each pair of composable 1-cells $f: A \rightarrow B$, $g: B \rightarrow C$ of \mathcal{S} , we have a 3-cell of \mathcal{T} :

$$\begin{array}{ccc} Fg.Ff & \xRightarrow{\chi_{f,g}^F} & F(gf) \\ \alpha_g \cdot \alpha_f \Downarrow & \Downarrow \Pi_{f,g}^\alpha & \Downarrow \alpha_{gf} \\ Gg.Gf & \xRightarrow{\chi_{f,g}^G} & G(gf) \end{array}).$$

These data are subject to the following axioms:

(IA1) For each 1-cell $f: A \rightarrow B$ of \mathcal{S} , the following pasting equality holds:

$$\begin{array}{c}
\begin{array}{ccccc}
& & FI_B.Ff \xrightarrow{\chi^F} F(I_B.f) & & \\
& \nearrow \iota^F.1 & & \searrow F\iota & \\
I_{FB}.Ff & & & & Ff \\
\downarrow 1.\alpha_f & \searrow \iota & \Downarrow \gamma^F & & \downarrow \alpha_f \\
I_{GB}.Gf & \cong & Ff & = & Ff \\
& \searrow \iota & \downarrow \alpha_f & = & \downarrow \alpha_f \\
& & Gf & = & Gf
\end{array} \\
= \begin{array}{ccccc}
& & FI_B.Ff \xrightarrow{\chi^F} F(I_B.f) & & \\
& \nearrow \iota^F.1 & \downarrow \alpha_{I_B} \cdot \alpha_f & \searrow \alpha_{I_B}.f & \\
I_{FB}.Ff & \xrightarrow{\overline{M^{\alpha.1}}} & GI_B.Gf \xrightarrow{\chi^G} G(I_B.f) & & Ff \\
\downarrow 1.\alpha_f & \nearrow \iota^G.1 & \downarrow \alpha_{I_B} & \searrow G\iota & \downarrow \alpha_f \\
I_{GB}.Gf & & & & Gf \\
& \searrow \iota & \Downarrow \gamma^G & & \\
& & Gf & = & Gf
\end{array}
\end{array}$$

(IA2) For each 1-cell $f: A \rightarrow B$ of \mathcal{S} , the following pasting equality holds:

$$\begin{array}{ccccc}
& & Ff.FI_B \xrightarrow{\chi^F} F(f.I_B) & & \\
& \nearrow 1.\iota^F & & \searrow F\tau & \\
Ff.I_{FB} & & & & Ff \\
\downarrow \alpha_f.1 & \searrow \tau & \Downarrow \delta^F & & \downarrow \alpha_f \\
Gf.I_{GB} & \cong & Ff & = & Ff \\
& \searrow \tau & \downarrow \alpha_f & = & \downarrow \alpha_f \\
& & Gf & = & Gf
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccccc}
& & Ff.FI_B & \xrightarrow{\chi^F} & F(f.I_B) \\
& \nearrow 1.t^F & \parallel \alpha_f.\alpha_{I_B} & \Downarrow \Pi^\alpha & \parallel \alpha_f.I_B \\
& & Ff.I_{FB} & \xrightarrow{1.M^\alpha} & Gf.GI_B & \xrightarrow{\chi^G} & G(f.I_B) & \Downarrow \alpha_\tau & Ff \\
& \parallel \alpha_f.1 & \parallel 1.t^G & & \parallel G\tau & \parallel \alpha_f \\
& & Gf.I_{GB} & \xrightarrow{\tau} & Gf & \xrightarrow{\tau} & Gf
\end{array} \\
= \\
\begin{array}{ccccc}
& & Ff.FI_B & \xrightarrow{\chi^F} & F(f.I_B) \\
& \nearrow 1.t^F & \parallel \alpha_f.\alpha_{I_B} & \Downarrow \Pi^\alpha & \parallel \alpha_f.I_B \\
& & Ff.I_{FB} & \xrightarrow{1.M^\alpha} & Gf.GI_B & \xrightarrow{\chi^G} & G(f.I_B) & \Downarrow \alpha_\tau & Ff \\
& \parallel \alpha_f.1 & \parallel 1.t^G & & \parallel G\tau & \parallel \alpha_f \\
& & Gf.I_{GB} & \xrightarrow{\tau} & Gf & \xrightarrow{\tau} & Gf
\end{array}
\end{array}$$

(IA3) For each triple f, g, h of composable 1-cells of \mathcal{S} , the following pasting equality holds:

$$\begin{array}{c}
\begin{array}{ccccc}
& & F(hg).Ff & \xrightarrow{\chi^F} & F((hg)f) \\
& \nearrow \chi^F.1 & \parallel \omega^F & \searrow Fa & \\
(Fh.Fg).Ff & \xrightarrow{a} & Fh.(Fg.Ff) & \xrightarrow{1.\chi^F} & Fh.F(gf) & \xrightarrow{\chi^F} & F(h(gf)) \\
\parallel (\alpha_h.\alpha_g).\alpha_f & \parallel a & \parallel \alpha_h.(\alpha_g.\alpha_f) & \parallel \alpha_h.\alpha_{gf} & \parallel \alpha_h(gf) \\
(Gh.Gg).Gf & \cong & Fh.(Fg.Ff) & \xrightarrow{1.\chi^F} & Fh.F(gf) & \xrightarrow{\chi^F} & F(h(gf)) \\
& \searrow a & \parallel \alpha_h.(\alpha_g.\alpha_f) & \parallel \alpha_h.\alpha_{gf} & \parallel \alpha_h(gf) \\
& & Gh.(Gg.Gf) & \xrightarrow{1.\chi^G} & Gh.G(gf)
\end{array} \\
= \\
\begin{array}{ccccc}
& & F(hg).Ff & \xrightarrow{\chi^F} & F((hg)f) \\
& \nearrow \chi^F.1 & \parallel \alpha_{hg}.\alpha_f & \parallel \Pi^\alpha & \parallel \alpha_{(hg)f} \\
(Fh.Fg).Ff & \xrightarrow{\Pi^\alpha.1} & G(hg).Gf & \xrightarrow{\chi^G} & G((hg)f) & \xrightarrow{Ga} & F(h(gf)) \\
\parallel (\alpha_h.\alpha_g).\alpha_f & \parallel \chi^G.1 & \parallel \omega^G & \parallel \alpha_a & \parallel \alpha_{h(gf)} \\
(Gh.Gg).Gf & \xrightarrow{a} & Gh.(Gg.Gf) & \xrightarrow{1.\chi^G} & Gh.G(gf)
\end{array}
\end{array}$$

Now, just as with oplax transformations between homomorphisms of bicategories, these oplax icons fail to admit a well-defined composition along 0-cell boundaries, and in order to remedy this we must restrict our attention to a suitable subclass. In fact, there are two such subclasses:

Definition 10. Let $F, G: \mathcal{S} \rightarrow \mathcal{T}$ be lax homomorphisms. Then:

- An **ico-icon** $\alpha: F \Rightarrow G$ is an oplax icon α for which each 2-cell $\alpha_f: Ff \Rightarrow Gf$ is an identity (it is an identity components oplax icon);
- A **pseudo-icon** $\alpha: F \Rrightarrow G$ is an oplax icon α for which each 3-cell α_θ , M_A^α and $\Pi_{f,g}^\alpha$ is invertible.

The ico-icons we have already met:

Proposition 11. Let $F, G: \mathcal{S} \rightarrow \mathcal{T}$ be lax homomorphisms. Then there is a bijection between the set of 2-cells $\alpha: F \Rightarrow G$ of \mathbf{Tricat}_2 and the set of oplax icons $\alpha: F \Rightarrow G$ for which each component $\alpha_f: Ff \Rightarrow Gf$ is an identity 2-cell.

On the other hand, a pseudo-icon is precisely what one gets by applying a naive dimension-raising to the notion of bicategorical icon: namely, inserting an invertible 3-cell into every diagram of 2-cells which previously commuted on the nose, and then subjecting all of this to an appropriate collection of axioms.

Now, these two subclasses of the oplax icons – the ico-icons and the pseudo-icons – will provide the respective vertical and horizontal 2-cells of \mathbf{Tricat}_3 . It remains only to introduce the 3-cells:

Definition 12. Let there be given pseudo-icons $\alpha: F \Rrightarrow G$ and $\beta: F' \Rrightarrow G'$, and ico-icons $\gamma: F \Rightarrow F'$ and $\delta: G \Rightarrow G'$. A **pseudo-icon modification**

$$\begin{array}{ccc} F & \xRightarrow{\alpha} & G \\ \gamma \Downarrow & \Downarrow \Gamma & \Downarrow \delta \\ F' & \xRightarrow{\beta} & G' \end{array}$$

now consists in the following datum:

(MD1) For each A, B in \mathcal{S} , a cubical modification (cf. Definition 7)

$$\begin{array}{ccc} F_{A,B} & \xRightarrow{\alpha_{A,B}} & G_{A,B} \\ \gamma_{A,B} \Downarrow & \Downarrow \Gamma_{A,B} & \Downarrow \delta_{A,B} \\ F'_{A,B} & \xRightarrow{\beta_{A,B}} & G'_{A,B} \end{array}$$

(so in particular, for each 1-cell $f: A \rightarrow B$ of \mathcal{S} , we have a 3-cell $\Gamma_f: \alpha_f \Rrightarrow \beta_f$ of \mathcal{T});

subject to the following axioms:

(MA1) For each object A of \mathcal{S} , the following pasting equality holds:

$$\begin{array}{ccc}
 & FI_A \xrightarrow{\alpha_{IA}} GI_A & \\
 \iota_A^F \nearrow & \Downarrow \Gamma_{IA} & \searrow \iota_A^G \\
 I_{FA} & \xrightarrow{\beta_{IA}} F'I_A & \xrightarrow{\beta_{IA}} G'I_A \\
 \Downarrow M_A^\gamma & \Downarrow M_A^\beta & \Downarrow M_A^\delta \\
 I_{F'A} & \xrightarrow{\beta_{IA}} I_{G'A} & \xrightarrow{\beta_{IA}} I_{G'A}
 \end{array} = \begin{array}{ccc}
 & FI_A \xrightarrow{\alpha_{IA}} GI_A & \\
 \iota_A^F \nearrow & \Downarrow M_A^\alpha & \searrow \iota_A^G \\
 I_{FA} & \xrightarrow{\beta_{IA}} I_{GA} & \xrightarrow{\beta_{IA}} G'I_A \\
 \Downarrow M_A^\gamma & \Downarrow M_A^\delta & \Downarrow M_A^\delta \\
 I_{F'A} & \xrightarrow{\beta_{IA}} I_{G'A} & \xrightarrow{\beta_{IA}} I_{G'A}
 \end{array}$$

(MA2) For each pair of composable 1-cells $f: A \rightarrow B$, $g: B \rightarrow C$ of \mathcal{S} , the following pasting equality holds:

$$\begin{array}{ccc}
 & F(gf) \xrightarrow{\alpha_{gf}} G(gf) & \\
 \chi_{f,g}^F \nearrow & \Downarrow \Gamma_{gf} & \searrow \chi_{f,g}^G \\
 Fg.Ff & \xrightarrow{\beta_{gf}} F'(gf) & \xrightarrow{\beta_{gf}} G'(gf) \\
 \Downarrow \Pi_{f,g}^\gamma & \Downarrow \Pi_{f,g}^\beta & \Downarrow \Pi_{f,g}^\delta \\
 F'g.F'f & \xrightarrow{\beta_{gf}} G'g.G'f & \xrightarrow{\beta_{gf}} G'g.G'f
 \end{array} = \begin{array}{ccc}
 & F(gf) \xrightarrow{\alpha_{gf}} G(gf) & \\
 \chi_{f,g}^F \nearrow & \Downarrow \Pi_{f,g}^\alpha & \searrow \chi_{f,g}^G \\
 Fg.Ff & \xrightarrow{\alpha_g \cdot \alpha_f} Gg.Gf & \xrightarrow{\alpha_g \cdot \alpha_f} G'(gf) \\
 \Downarrow \Gamma_g \cdot \Gamma_f & \Downarrow \Pi_{f,g}^\delta & \Downarrow \Pi_{f,g}^\delta \\
 F'g.F'f & \xrightarrow{\beta_g \cdot \beta_f} G'g.G'f & \xrightarrow{\beta_g \cdot \beta_f} G'g.G'f
 \end{array}$$

We are now ready to start constructing \mathfrak{Tricat}_3 . We begin with the local structure:

Proposition 13. *Let \mathcal{S} and \mathcal{T} be tricategories. Then the lax homomorphisms, ico-icons, pseudo-icons and pseudo-icon modifications from \mathcal{S} to \mathcal{T} form a weak double category $\mathfrak{Icon}(\mathcal{S}, \mathcal{T})$.*

Proof. Underlying each lax homomorphism, ico-icon, pseudo-icon or pseudo-icon modification is an indexed family of homomorphisms of bicategories, bicategorical icons, pseudo-natural transformations, or cubical modifications, respectively: thus our approach will be to lift the compositional structure from the double categories $\mathfrak{Hom}(\mathcal{C}, \mathcal{D})$ as defined on page 19.

We begin with the vertical structure of $\mathfrak{Icon}(\mathcal{S}, \mathcal{T})$. We have already seen in Section 2 that the lax homomorphisms and ico-icons from \mathcal{S} to \mathcal{T} form a category; we must show the same is true of the oplax icons and the icon modifications between them. So given an oplax icon $\alpha: F \rightrightarrows G$, the unit icon modification $\text{id}_\alpha: \alpha \Rightarrow \alpha$ is given by the identity family of modifications

$$(\text{id}_\alpha)_{A,B} = \text{id}_{\alpha_{A,B}}: \alpha_{A,B} \Rightarrow \alpha_{A,B};$$

the axioms (MA1) and (MA2) are clear, since every occurrence of Γ reduces to an identity 3-cell. Next, given icon modifications $\Gamma: \alpha \Rightarrow \beta$ and $\Delta: \beta \Rightarrow \gamma$, their

vertical composite $\Delta\Gamma: \alpha \Rightarrow \gamma$ is given by composing their underlying families of modifications:

$$(\Delta\Gamma)_{A,B} = \Delta_{A,B} \cdot \Gamma_{A,B}: \alpha_{A,B} \Rightarrow \gamma_{A,B}.$$

Now the axioms (MA1) and (MA2) follow from an application of the corresponding axiom for Δ followed by the corresponding axiom for Γ . Associativity and unitality of this composition follow from that of vertical composition of modifications.

We next describe the horizontal structure of $\mathfrak{Icon}(\mathcal{S}, \mathcal{T})$, beginning with the identities functor $I_{(-)}$. Given a lax homomorphism $F: \mathcal{S} \rightarrow \mathcal{T}$, the identity pseudo-icon $I_F: F \Rightarrow F$ has (ID1) given by the family:

$$(I_F)_{A,B} = \text{id}_{F_{A,B}}: F_{A,B} \Rightarrow F_{A,B};$$

whilst $M_A^{I_F}$ and $\Pi_{A,B,C}^{I_F}$ are given by unnamed coherence isomorphisms in the hom-bicategories of \mathcal{T} , whilst for an ico-icon $\alpha: F \Rightarrow G$, the identity icon modification $I_\alpha: I_F \Rightarrow I_G$ has (MD1) given by the identity family of 3-cells

$$\text{id}_{\text{id}_{Ff}}: \text{id}_{Ff} \Rightarrow \text{id}_{Gf}.$$

Each of the axioms (IA1)–(IA3) for I_F and (MA1)–(MA2) for I_α now asserts that some 3-cell is equal to itself when pasted with such unnamed coherence cells, and this follows from coherence for bicategories. The functoriality of $I_{(-)}$ is immediate.

We now describe the horizontal composition functor for this double category. Given pseudo-icons $\alpha: F \Rightarrow G$ and $\beta: G \Rightarrow H$, we define an pseudo-icon $\beta\alpha: F \Rightarrow H$ as follows:

$$(\text{ID1}) \quad (\beta\alpha)_{A,B} = \beta_{A,B} \cdot \alpha_{A,B}: F_{A,B} \Rightarrow H_{A,B};$$

$$(\text{ID2}) \quad M_A^{\beta\alpha} \text{ is the pasting:}$$

$$\begin{array}{ccc} I_{FA} & \xRightarrow{\iota_A^F} & FI_A \\ \parallel & \Downarrow M_A^\alpha & \Downarrow \alpha_{I_A} \\ I_{GA} & \xRightarrow{\iota_A^G} & GI_A \\ \parallel & \Downarrow M_A^\beta & \Downarrow \beta_{I_A} \\ I_{HA} & \xRightarrow{\iota_A^H} & HI_A; \end{array}$$

$$(\text{ID3}) \quad \Pi_{A,B,C}^\alpha \text{ is the pasting:}$$

$$\begin{array}{ccccc} & & F(-) \otimes F(?) & \xRightarrow{\chi^F} & F((-) \otimes (?)) \\ & \searrow & \downarrow \alpha_{(-)} \otimes \alpha_{(?)} & \Downarrow \Pi_{A,B,C}^\alpha & \downarrow \alpha_{((-) \otimes (?))} \\ & & \cong G(-) \otimes G(?) & \xRightarrow{\chi^G} & G((-) \otimes (?)) \\ & \swarrow & \downarrow \beta_{(-)} \otimes \beta_{(?)} & \Downarrow \Pi_{A,B,C}^\beta & \downarrow \beta_{((-) \otimes (?))} \\ & & H(-) \otimes H(?) & \xRightarrow{\chi^H} & H((-) \otimes (?)). \end{array}$$

(A curved arrow labeled $(\beta\alpha)_{(-)} \otimes (\beta\alpha)_{(?)}$ points from the top-left $F(-) \otimes F(?)$ to the bottom-left $H(-) \otimes H(?)$.)

Showing that these data satisfy axioms (IA1)–(IA3) is almost as simple as placing the corresponding diagrams for β and α alongside each other; the only slight complication arises from the presence of the barred 3-cells, which require us to prove some auxiliary results. For instance, in order to prove (IA1) we must first show that:

$$\begin{array}{ccc}
I_{FB}.Ff & \xrightarrow{\iota^F.1} & FI_B.Ff \\
\downarrow 1.\alpha_f & \Downarrow \overline{M^{\alpha.1}} & \downarrow \alpha_{I_B}.\alpha_f \\
\cong I_{GB}.Gf & \xrightarrow{\iota^G.1} & GI_B.Gf \cong \\
\downarrow 1.\beta_f & \Downarrow \overline{M^{\beta.1}} & \downarrow \beta_{I_B}.\beta_f \\
I_{HB}.Hf & \xrightarrow{\iota^H.1} & HI_B.Hf
\end{array}
\begin{array}{l}
1.(\beta\alpha)_f \quad \quad \quad (\beta\alpha)_{I_B}.(\beta\alpha)_f
\end{array}$$

$$= \begin{array}{ccc}
I_{FB}.Ff & \xrightarrow{\iota^F.1} & FI_B.Ff \\
\downarrow 1.(\beta\alpha)_f & \Downarrow \overline{M^{\beta\alpha.1}} & \downarrow (\beta\alpha)_{I_B}.(\beta\alpha)_f \\
I_{HB}.Hf & \xrightarrow{\iota^H.1} & HI_B.Hf;
\end{array}$$

holds; and similarly for (IA2) and (IA3). These derivations are straightforward bicategorical manipulations and left to the reader. Next we consider the horizontal composition of 2-cells in $\mathfrak{Icon}(\mathcal{S}, \mathcal{T})$. Next, given icon modifications

$$\begin{array}{ccccc}
F & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & H \\
\sigma \Downarrow & \Downarrow \Gamma & \Downarrow \tau & \Downarrow \Delta & \Downarrow v \\
F' & \xrightarrow{\alpha'} & G' & \xrightarrow{\beta'} & H'
\end{array}$$

we define the icon modification $\Delta * \Gamma: \beta\alpha \Rightarrow \beta'\alpha': F \Rightarrow H$ by taking

$$(\Delta * \Gamma)_{A,B} = \Delta_{A,B} * \Gamma_{A,B}: \beta_{A,B}\alpha_{A,B} \Rightarrow \beta'_{A,B}\alpha'_{A,B},$$

where $*$ on the right-hand side is horizontal composition of 2-cells in the weak double category $\mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{T}(FA, FB))$. In particular, for any 1-cell f of \mathcal{S} , the 3-cell $(\Delta * \Gamma)_f$ is given by the pasting

$$\begin{array}{ccccc}
Ff & \xrightarrow{\alpha_f} & Gf & \xrightarrow{\beta_f} & Hf, \\
\parallel & \Downarrow \Gamma_f & \parallel & \Downarrow \Delta_f & \parallel \\
Ff & \xrightarrow{\alpha'_f} & Gf & \xrightarrow{\beta'_f} & Hf
\end{array}$$

and thus (MA1) and (MA2) for $\Delta * \Gamma$ follow by placing the corresponding axioms for Γ and Δ beside each other, together with some very simple manipulation with unnamed coherence cells. Finally, note that the middle-four interchange law will hold in $\mathfrak{Icon}(\mathcal{S}, \mathcal{T})$ since it does in each double category $\mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{T}(FA, FB))$.

It remains only to give the unitality and associativity constraints \mathfrak{a} , \mathfrak{l} and \mathfrak{r} for our double category. So let there be given pseudo-icons $\alpha: F \Rightarrow G$, $\beta: G \Rightarrow H$ and $\gamma: H \Rightarrow K$. Then:

- The associativity constraint $\mathfrak{a}_{\alpha,\beta,\gamma}: (\gamma\beta)\alpha \Rightarrow \gamma(\beta\alpha)$ has component modification $(\mathfrak{a}_{\alpha,\beta,\gamma})_{A,B}$ given by the associativity constraint $\mathfrak{a}_{\alpha_{A,B},\beta_{A,B},\gamma_{A,B}}$ in the double category $\mathfrak{Hom}(\mathcal{S}(A,B), \mathcal{T}(FA, FB))$;
- The left unitality constraint $\mathfrak{l}_\alpha: \text{id}_G.\alpha \Rightarrow \alpha$ has component modification $(\mathfrak{l}_\alpha)_{A,B}$ given by the left unitality constraint $\mathfrak{l}_{\alpha_{A,B}}$ in $\mathfrak{Hom}(\mathcal{S}(A,B), \mathcal{T}(FA, FB))$;
- The right unitality constraint $\mathfrak{r}_\alpha: \alpha.\text{id}_G \Rightarrow \alpha$ has component modification $(\mathfrak{r}_\alpha)_{A,B}$ given by the right unitality constraint $\mathfrak{r}_{\alpha_{A,B}}$ in $\mathfrak{Hom}(\mathcal{S}(A,B), \mathcal{T}(FA, FB))$.

The naturality of these constraints in α , β and γ is inherited from the hom-double categories $\mathfrak{Hom}(\mathcal{S}(A,B), \mathcal{T}(FA, FB))$; and that these data satisfy the axioms (MA1) and (MA2) is also straightforward. In the case of $\mathfrak{a}_{\alpha,\beta,\gamma}$, for example, we see that $M_A^{\gamma(\beta\alpha)}$ and $\Pi_{f,g}^{\gamma(\beta\alpha)}$ can be obtained from $M_A^{(\gamma\beta)\alpha}$ and $\Pi_{f,g}^{(\gamma\beta)\alpha}$ by pasting with unnamed coherence isomorphisms; but the components of $\mathfrak{a}_{\alpha,\beta,\gamma}$ are built from the selfsame coherence isomorphisms, and so the result follows from the coherence theorem for bicategories. \square

In order for the double categories $\mathfrak{Icon}(\mathcal{S}, \mathcal{T})$ to provide homs for the locally double bicategory \mathfrak{Tricat}_3 , we must define top level composition and identity homomorphisms. The identity homomorphism

$$\lrcorner I_{\mathcal{T}} \lrcorner : 1 \rightarrow \mathfrak{Icon}(\mathcal{T}, \mathcal{T})$$

is straightforward; it sends the unique object of 1 to the identity lax homomorphism $\text{id}_{\mathcal{T}}: \mathcal{T} \rightarrow \mathcal{T}$, the unique vertical 1-cell to the identity ico-icon on $\text{id}_{\mathcal{T}}$; the unique horizontal 1-cell to the identity pseudo-icon on $\text{id}_{\mathcal{T}}$; and the unique 2-cell to the identity icon modification on this. The coherence data for this homomorphism arises from unitality constraints in $\mathfrak{Icon}(\mathcal{T}, \mathcal{T})$, and so the homomorphism axioms follow from coherence for bicategories. Let us now consider the composition homomorphism

$$\otimes: \mathfrak{Icon}(\mathcal{T}, \mathcal{U}) \times \mathfrak{Icon}(\mathcal{S}, \mathcal{T}) \rightarrow \mathfrak{Icon}(\mathcal{S}, \mathcal{U}).$$

The general approach will be similar to that adopted in the proof of Proposition 13. There, we defined the compositional structure on $\mathfrak{Icon}(\mathcal{S}, \mathcal{T})$ by lifting it from the weak double categories $\mathfrak{Hom}(\mathcal{S}(A,B), \mathcal{T}(FA, FB))$: here, we will define \otimes by lifting the composition homomorphisms

$$\begin{aligned} \mathfrak{Hom}(\mathcal{T}(FA, FB), \mathcal{U}(GFA, GFB)) \times \mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{T}(FA, FB)) \\ \rightarrow \mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{U}(GFA, GFB)) \end{aligned}$$

which we obtain from the locally double bicategory \mathfrak{Bicat} of Proposition 6.

In detail, \otimes is given as follows. On objects and vertical 1-cells, it is given by the composition law for \mathfrak{Tricat}_2 . On horizontal 1-cells, we consider pseudo-icons $\alpha: F \Rightarrow F': \mathcal{S} \rightarrow \mathcal{T}$ and $\beta: G \Rightarrow G': \mathcal{T} \rightarrow \mathcal{U}$, for which the composite pseudo-icon $\beta \otimes \alpha: GF \Rightarrow G'F'$ is given as follows:

(ID1) $(\beta \otimes \alpha)_{A,B} = \beta_{FA,FB} \otimes \alpha_{A,B}$, where \otimes on the right-hand side is one of the two canonical choices for composition of horizontal composition of pseudo-natural transformations; for concreteness let us take

$$(\beta \otimes \alpha)_f = GFf \xRightarrow{\beta_{Ff}} G'Ff \xRightarrow{G'\alpha_f} G'F'f$$

and

$$\begin{array}{c}
 GFf \xRightarrow{GF\theta} GFg \\
 \beta_{Ff} \Downarrow \quad \Downarrow \beta_{F\theta} \quad \Downarrow \beta_{Fg} \\
 G'Ff \xRightarrow{G'F\theta} G'Fg \\
 G'\alpha_f \Downarrow \quad \Downarrow G'\alpha_\theta \quad \Downarrow G'\alpha_g \\
 G'F'f \xRightarrow{G'F'\theta} G'F'g;
 \end{array}
 \quad (\beta \otimes \alpha)_\theta =$$

(ID2) $M_A^{\beta \otimes \alpha}$ is the following 3-cell:

$$\begin{array}{ccccc}
 I_{GFA} & \xRightarrow{\iota^G} & GI_{FA} & \xRightarrow{G\iota^F} & GFIA \\
 \parallel & & \Downarrow M_{FA}^\beta & \Downarrow \beta_{I_{FA}} & \Downarrow \beta_{\iota^F} & \Downarrow \beta_{FI_A} \\
 I_{G'FA} & \xRightarrow{\iota^{G'}} & G'I_{FA} & \xRightarrow{G'\iota^F} & G'FI_A \\
 \parallel & = & \parallel & & \Downarrow \overline{G'M_A^\alpha} & \Downarrow G'\alpha_{I_A} \\
 I_{G'F'A} & \xRightarrow{\iota^{G'}} & G'I_{F'A} & \xRightarrow{G'\iota^{F'}} & G'F'I_A;
 \end{array}$$

(ID3) $\Pi_{A,B,C}^{\beta \otimes \alpha}$ is the pseudo-natural transformation with the following components:

$$\begin{array}{c}
 GFg, GFf \xRightarrow{\chi^G} G(Fg.Ff) \xRightarrow{G\chi^F} GF(gf) \\
 \beta_{Fg}, \beta_{Ff} \Downarrow \quad \Downarrow \Pi_{Ff,Fg}^\beta \quad \Downarrow \beta_{Fg.Ff} \quad \Downarrow \beta_{\chi^F} \quad \Downarrow \beta_{F(gf)} \\
 (\beta \otimes \alpha)_g, (\beta \otimes \alpha)_f \cong G'Fg, G'Ff \xRightarrow{\chi^{G'}} G'(Fg.Ff) \xRightarrow{G'\chi^F} G'F(gf) \\
 G'\alpha_g, G'\alpha_f \Downarrow \quad \cong \quad G'(\alpha_g, \alpha_f) \quad \Downarrow \overline{G'\Pi_{f,g}^\alpha} \quad \Downarrow G'\alpha_{gf} \\
 G'F'g, G'F'f \xRightarrow{\chi^{G'}} G'(F'g.F'f) \xRightarrow{G'\chi^{F'}} G'F'(gf).
 \end{array}$$

The proof that these data satisfy axioms (IA1)–(IA3) consists once again in building large cubes or hexagonal prisms from smaller ones, together with some simple manipulation with unnamed coherence cells: and once again, we leave this task to the reader.

Finally we must give the action of \otimes on 2-cells. Given two such:

$$\begin{array}{ccc}
 F \xRightarrow{\alpha} F' & & G \xRightarrow{\beta} G' \\
 \sigma \Downarrow & \Downarrow \Gamma & \Downarrow \sigma' \\
 H \xRightarrow{\gamma} H' & & K \xRightarrow{\delta} K'
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 G \xRightarrow{\beta} G' & & \\
 \tau \Downarrow & \Downarrow \Delta & \Downarrow \tau' \\
 K \xRightarrow{\delta} K' & &
 \end{array}$$

in $\mathfrak{Icon}(\mathcal{S}, \mathcal{T})$ and $\mathfrak{Icon}(\mathcal{T}, \mathcal{U})$ respectively, we define their composite $\Delta \otimes \Gamma: \beta \otimes \alpha \Rightarrow \delta \otimes \gamma$ to be given by horizontally composing the underlying families of cubical modifications in the locally double bicategory \mathfrak{Bicat} :

$$(\Delta \otimes \Gamma)_{A,B} = \Gamma_{FA,FB} \otimes \Delta_{A,B}: \beta_{FA,FB} \otimes \alpha_{A,B} \Rightarrow \delta_{FA,FB} \otimes \gamma_{A,B}.$$

So in particular, for any 1-cell $f: A \rightarrow B$ of \mathcal{S} , we have $(\Delta \otimes \Gamma)_f$ given by the following pasting

$$\begin{array}{ccccc} GFf & \xRightarrow{\beta_{Ff}} & G'Ff & \xRightarrow{G'\alpha_f} & G'F'f \\ \parallel & \Downarrow \Delta_{Ff} & \parallel & \Downarrow \tau'_{\alpha_f} & \parallel \\ KFf & \xRightarrow{\delta_{Ff}} & K'Ff & \xRightarrow{K'\alpha_f} & K'F'f \\ \parallel & = & \parallel & \Downarrow K'\Gamma_f & \parallel \\ KHf & \xRightarrow{\delta_{Hf}} & K'Hf & \xRightarrow{K'\gamma_f} & K'H'f. \end{array}$$

Proving axioms (MA1) and (MA2) for this data amounts to constructing a further succession of pasting equalities which traverse the interior of a $2 \times 2 \times 2$ cube, using:

- the corresponding axioms (MA1) or (MA2) for Δ and Γ ,
- the cubical modification axioms for the components of Δ ,
- the icon axioms for the components of τ' ,
- the pseudo-natural transformation axioms for the components of δ ,
- and some further calculus with unnamed coherence isomorphisms.

We leave this task to the reader. Functoriality of this composition with respect to vertical composition is inherited from that of horizontal composition of modifications in \mathfrak{Bicat} .

It remains to exhibit the pseudo-functoriality constraints for \otimes ; so let there be given lax homomorphisms and pseudo-icons

$$\begin{array}{ccccc} & F & & G & \\ & \downarrow \alpha & & \downarrow \beta & \\ \mathcal{S} & \xrightarrow{F'} & \mathcal{T} & \xrightarrow{G'} & \mathcal{U} \\ & \downarrow \gamma & & \downarrow \delta & \\ & F'' & & G'' & \end{array}$$

We must exhibit invertible globular icon modifications

$$i_{(G,F)}: \text{id}_{GF} \Rightarrow \text{id}_G \otimes \text{id}_F: GF \Rightarrow GF \quad \text{and} \quad m_{(\beta,\alpha),(\delta,\gamma)}: (\delta \otimes \gamma)(\beta \otimes \alpha) \Rightarrow (\delta \beta) \otimes (\gamma \alpha);$$

whose respective (A, B) -components we take to be the invertible modifications witnessing pseudo-functoriality of horizontal composition in the following diagram of

homomorphisms and pseudo-natural transformations:

$$\begin{array}{ccccc}
& F_{A,B} & & G_{FA,FB} & \\
& \Downarrow \alpha_{A,B} & & \Downarrow \beta_{FA,FB} & \\
\mathcal{S}(A,B) & \xrightarrow{F'_{A,B}} & \mathcal{T}(FA,FB) & \xrightarrow{G'_{FA,FB}} & \mathcal{U}(GFA,GFB) \\
& \Downarrow \gamma_{A,B} & & \Downarrow \delta_{FA,FB} & \\
& F''_{A,B} & & G''_{FA,FB} &
\end{array}$$

We must check that these data satisfy axioms (MA1) and (MA2). The proof is straightforward manipulation using the pseudo-naturality axioms for δ and the modification axioms for Π^δ . Finally, the naturality of the maps $m_{(\beta,\alpha),(\delta,\gamma)}$ in all variables follows componentwise; as do the coherence axioms which m and i must satisfy.

Theorem 14. *The double categories $\mathfrak{Icon}(\mathcal{S}, \mathcal{T})$ together with the identity and composition homomorphisms defined above provide data for a locally double bicategory \mathfrak{Tricat}_3 with 0-cells being tricategories; 1-cells, lax homomorphisms; vertical 2-cells, ico-icons; horizontal 2-cells, pseudo-icons; and 3-cells, icon modifications.*

Proof. All that remains is to give the associativity and unitality constraints a , l and r for \mathfrak{Tricat}_3 , and check the triangle and pentagon axioms. At the level of 1-cells and vertical 2-cells, these are the corresponding constraints from \mathbf{Tricat}_2 ; whilst at the level of horizontal 2-cells and 3-cells, suppose we are given trihomomorphisms and pseudo-icons

$$\begin{array}{ccccc}
& F & & G & & H \\
& \Downarrow \alpha & & \Downarrow \beta & & \Downarrow \gamma \\
\mathcal{R} & \xrightarrow{F'} & \mathcal{S} & \xrightarrow{G'} & \mathcal{T} & \xrightarrow{H'} & \mathcal{U}
\end{array}$$

Then we must give an icon modification

$$\begin{array}{ccc}
(HG)F & \xRightarrow{(\gamma \otimes \beta) \otimes \alpha} & (H'G')F' \\
\Downarrow a_{F,G,H} & & \Downarrow a_{F',G',H'} \\
H(GF) & \xRightarrow{\gamma \otimes (\beta \otimes \alpha)} & H'(G'F')
\end{array}$$

where $a_{F,G,H}$ and $a_{F',G',H'}$ are the corresponding constraints from \mathbf{Tricat}_2 . So we set the (A, B) th component of this icon modification to be the cubical modification which provides the associativity constraint for the composition

$$\begin{array}{ccccc}
\mathcal{R}(A,B) & \xrightarrow{F_{A,B}} & \mathcal{S}(FA,FB) & \xrightarrow{G_{FA,FB}} & \mathcal{T}(GFA,GFB) & \xrightarrow{H_{GFA,GFB}} & \mathcal{U}(HGFA,HGFB) \\
& \Downarrow \alpha_{A,B} & & \Downarrow \beta_{FA,FB} & & \Downarrow \gamma_{GFA,GFB} & \\
& F'_{A,B} & & G'_{FA,FB} & & H'_{GFA,GFB} &
\end{array}$$

in the locally double bicategory \mathfrak{Bicat} . We must check that these satisfy the data satisfy the axioms for an icon modification; let us do only (MA2), since (MA1) follows identically. We first observe that the 3-cells $\Pi_A^{(\gamma \otimes \beta) \otimes \alpha}$ and $\Pi_A^{\gamma \otimes (\beta \otimes \alpha)}$ are

obtained by pasting together what is essentially the same 3×3 diagram of 3-cells, and some trivial calculus with unnamed coherence cells shows that they are precisely the same diagram, modulo rewriting of the boundary, so that the latter 3-cell may be obtained from the former by pasting with unnamed coherence cells. But this is precisely the content of axiom (MA2).

Finally, we must check that these icon modifications $a_{\alpha,\beta,\gamma}$ are natural in α , β and γ , and satisfy the pentagon and triangle equalities. Each of these follows componentwise from the corresponding facts in \mathfrak{Bicat} . \square

5 From locally double bicategories to tricategories

As we have seen, the structure which captures most precisely the compositional calculus of tricategorical icons is a locally double bicategory. Nonetheless, there may be occasions when we are prepared to sacrifice some of this precision in order to work with a more familiar structure like a tricategory: and so we would like to extract a description of a tricategory \mathbf{Tricat}_3 from our description of \mathfrak{Tricat}_3 . Happily, there are general principles at work which allow us to do just that.

In this Section, we will show how every sufficiently-well behaved locally double bicategory \mathfrak{C} gives rise to a tricategory. This tricategory will have the same 0- and 1-cells as \mathfrak{C} ; as 2-cells, the horizontal 2-cells of \mathfrak{C} ; and as 3-cells, the globular 3-cells of \mathfrak{C} . The main point of interest is the construction of the tricategorical associativity constraints, which are to be given by *horizontal* 2-cells of \mathfrak{C} . Since the associativity constraints in \mathfrak{C} are given by *vertical* 2-cells, we will need some kind of linkage between the two types of 2-cell in order to proceed.

Definition 15. A weak double category \mathfrak{C} is said to be **fibrant** if the functor $(s, t): \mathcal{C}_1 \rightarrow \mathcal{C}_0 \times \mathcal{C}_0$ is an isofibration.

Recall here that a functor $F: \mathcal{A} \rightarrow \mathcal{B}$ between categories is an *isofibration* if whenever we have a object $a \in \mathcal{A}$ and isomorphism $\phi: Fa \rightarrow b$ in \mathcal{B} , there exists an object $c \in \mathcal{A}$ and isomorphism $\theta: a \rightarrow c$ such that $Fc = b$ and $F\theta = \phi$. Thus a weak double category \mathfrak{C} is fibrant just when every diagram like (a) below has a filler like (b) for which the 2-cell θ is invertible as an arrow of \mathcal{C}_1 :

$$(a) \quad \begin{array}{ccc} x & & x' \\ f \downarrow & & \downarrow g \\ y & \xrightarrow{k} & y' \end{array} \quad \rightsquigarrow \quad (b) \quad \begin{array}{ccc} x & \xrightarrow{h} & x' \\ f \downarrow & \Downarrow \theta & \downarrow g \\ y & \xrightarrow{k} & y' \end{array}$$

Thus fibrancy is precisely the property which [9] refers to as *horizontal invariance*. We may reformulate this property in various useful ways, and since detailed accounts of this process may be found in [6] or [10], we restrict ourselves here to recording those equivalent formulations which will be useful to us.

For the first, we consider the case of the above filling condition where g and k are both identities: given a vertical map $f: x \rightarrow y$ of \mathfrak{C} , it asserts the existence of a

horizontal 1-cell \overline{f} and a 2-cell ϵ_f fitting into the diagram:

$$\begin{array}{ccc} x & \xrightarrow{\overline{f}} & y \\ f \downarrow & \Downarrow \epsilon_f & \downarrow \text{id}_y \\ y & \xrightarrow{I_y} & y. \end{array}$$

From this, we may define a further 2-cell η_f as the composite

$$\begin{array}{ccc} x & \xrightarrow{I_x} & x \\ \text{id}_x \downarrow & \Downarrow \eta_f & \downarrow f \\ x & \xrightarrow{\overline{f}} & y \end{array} \quad := \quad \begin{array}{ccc} x & \xrightarrow{I_x} & x \\ f \downarrow & \Downarrow I_f & \downarrow f \\ y & \xrightarrow{g} & y \\ f^{-1} \downarrow & \Downarrow \epsilon_f^{-1} & \downarrow \text{id}_y \\ x & \xrightarrow{\overline{f}} & y. \end{array}$$

Now the pair (η_f, ϵ_f) satisfy the triangle identities:

$$\epsilon_f \cdot \eta_f = I_f : I_x \Rightarrow I_y \quad \text{and} \quad \epsilon_f * \eta_f = (l^{-1}r)_{\overline{f}} : \overline{f} \cdot I_x \Rightarrow I_y \cdot \overline{f},$$

and so, in the terminology of [1], f and \overline{f} are *orthogonal companions*; which gives us the “only if” direction of:

Proposition 16. *A weak double category \mathfrak{C} is fibrant iff every vertical isomorphism has an orthogonal companion.*

For the “if” direction, suppose that we are given a diagram like (a); then we can complete it to a diagram like (b) by taking h to be $\overline{g^{-1}} \cdot (k \cdot \overline{f})$ and θ to be the 2-cell:

$$\overline{g^{-1}} \cdot (k \cdot \overline{f}) \xrightarrow{(I_g \cdot \epsilon_{g^{-1}}) * (\text{id}_k * \epsilon_f)} I_{y'} \cdot (k \cdot I_y) \xrightarrow{I_{y'} * r_k} I_{y'} \cdot k \xrightarrow{l_k} k.$$

Thus each of \mathfrak{Cat} , \mathfrak{Rng} and $\mathfrak{Span}(\mathcal{C})$ is a fibrant double category: for \mathfrak{Cat} , the horizontal companion of a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is the profunctor $\overline{F}(-, ?) = \mathcal{D}(-, F?)$; for \mathfrak{Rng} , the companion of a homomorphism $f : R \rightarrow S$ is S itself, viewed as a left S -, right R -module; and in $\mathfrak{Span}(\mathcal{C})$, the companion of a morphism $f : C \rightarrow D$ is the span $C \xleftarrow{\text{id}} C \xrightarrow{f} D$. Observe that in all of these examples, it is arbitrary vertical morphisms, and not just the isomorphisms, which have companions: such weak double categories are essentially the *pro-arrow equipments* of [19, 20]. A more detailed analysis of this correspondence may be found in Appendix C of [17].

Proposition 17. *Let \mathfrak{C} be a fibrant double category equipped with a choice of orthogonal companion for every vertical isomorphism. Then the assignment $f \mapsto \overline{f}$ underlies an identity-on-objects homomorphism of bicategories*

$$(\overline{}) : V^{iso}(\mathfrak{C}) \rightarrow H(\mathfrak{C}),$$

where $V^{\text{iso}}(\mathfrak{C})$ is the category of objects and vertical isomorphisms in \mathfrak{C} . Moreover, if we are given vertical isomorphisms $f: w \rightarrow y$ and $g: x \rightarrow z$ in \mathfrak{C} , then pasting with η_f and ϵ_g induces a bijection between the set of 2-cells of the form (c) and the set of 2-cells of the form (d):

$$(c) \quad \begin{array}{ccc} x & \xrightarrow{h} & x' \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ y & \xrightarrow{k} & y' \end{array} \quad \text{and} \quad (d) \quad \begin{array}{ccc} x & \xrightarrow{\bar{g}.h} & y' \\ \text{id}_x \downarrow & \Downarrow \bar{\alpha} & \downarrow \text{id}_{y'} \\ x & \xrightarrow{k.\bar{f}} & y' \end{array};$$

and $\bar{\alpha}$ is invertible as an arrow of \mathcal{C}_1 just when α is. Furthermore, these bijections satisfy four evident axioms expressing their functoriality with respect to vertical and horizontal composition of 2-cells.

The proof is straightforward manipulation, and it is not hard to prove a converse – namely, that from a homomorphism of bicategories $(\bar{}): V^{\text{iso}}(\mathfrak{C}) \rightarrow H(\mathfrak{C})$ and a bijective assignment $\alpha \mapsto \bar{\alpha}$ on 2-cells satisfying the four functoriality axioms, one may define a choice of orthogonal companion for every vertical isomorphism. A detailed proof may be found in the pages leading up to Theorem 3.28 of [6].

Definition 18. The 2-category \mathbf{DbCat}_f has objects being fibrant weak double categories equipped with a choice of orthogonal companions; as 1-cells, the homomorphisms between the underlying double categories; and as 2-cells, the vertical transformations between them.

One may reasonably ask why we do not require the 1-cells $F: \mathfrak{C} \rightarrow \mathfrak{D}$ of \mathbf{DbCat}_f to respect the choices of orthogonal companions in \mathfrak{C} and \mathfrak{D} . The reason is that, in fact, *any* homomorphism between objects of \mathbf{DbCat}_f will automatically respect these choices in a unique way. To make this explicit, let us say that a homomorphism $F: \mathfrak{C} \rightarrow \mathfrak{D}$ between objects of \mathbf{DbCat}_f is a **fibrant homomorphism** if, for every invertible vertical 1-cell $f: x \rightarrow y$ of \mathfrak{C} , there is given an invertible globular 2-cell

$$\mu_f: F(\bar{f}) \Rightarrow \overline{Ff}: Fx \dashv\dashv Fy$$

of \mathfrak{D} , subject to three axioms. The first two equate, respectively, the two possible 2-cells in \mathfrak{D} from I_{Fx} to $\overline{F(\text{id}_x)}$; and from $F(\bar{g}) \cdot F(\bar{f})$ to $\overline{F(gf)}$. The third axiom concerns a 2-cell α of the type (c) above, and equates the two globular 2-cells

$$\begin{array}{ccc} \begin{array}{ccc} Fx & \xrightarrow{Fh} & x' \\ \downarrow Ff & \Downarrow F(\bar{\alpha}) & \downarrow F(\bar{g}) \\ Fy & \xrightarrow{Fk} & y' \end{array} & \text{and} & \begin{array}{ccc} Fx & \xrightarrow{Fh} & x' \\ \downarrow Ff & \Downarrow \overline{F\alpha} & \downarrow \overline{Fg} \\ Fy & \xrightarrow{Fk} & y' \end{array} \end{array}$$

Now, given any homomorphism $F: \mathfrak{C} \rightarrow \mathfrak{D}$ between objects of \mathbf{DbCat}_f , we may make it into a fibrant homomorphism as follows. Given an invertible vertical arrow $f: x \rightarrow y$ of \mathfrak{C} , we can consider the globular 2-cell

$$\overline{F\epsilon_f}: \overline{\text{id}_{Fy}} \cdot F\bar{f} \Rightarrow FI_y \cdot \overline{Ff}$$

of \mathfrak{D} ; and since both $\overline{\text{id}_{Fy}}$ and FI_y are isomorphic to I_{Fy} , we obtain from this a globular 2-cell $\mu_f: F\overline{f} \Rightarrow \overline{F}f$, which is easily checked to satisfy the three axioms. And in fact, this is the only possible structure of fibrant homomorphism on F : for given an arbitrary such structure, applying the third axiom to the 2-cells ϵ_f in \mathfrak{C} shows that the maps μ_f must, in fact, coincide with those defined above. A similar argument applies to the 2-cells of \mathbf{DbCat}_f .

A conceptual explanation of why this should be the case is that \mathbf{DbCat}_f is, in some sense, the 2-category of algebras for a particularly simple kind of 2-dimensional monad on \mathbf{DbCat} , the kind which [12] calls *pseudo-idempotent*: and such monads have the property that the forgetful functor from the 2-category of algebras and algebra pseudomorphisms to the underlying base 2-category is 2-fully faithful. The qualifier “in some sense” covers a slight wrinkle in this explanation: namely, that the 2-monad which gives rise to \mathbf{DbCat}_f lives not on \mathbf{DbCat} but on $\mathbf{DbCat}_{\text{str}}$, the 2-category of weak double categories and *strict* homomorphisms between them, so that making this argument rigorous would require a little more work.

Definition 19. We will say that a locally double bicategory is **locally fibrant** just when each of its hom-double categories is fibrant.

In particular, a monoidal double category is locally fibrant just when its underlying weak double category is fibrant, so that all of our examples of monoidal double categories are locally fibrant. The locally double bicategory \mathfrak{DbCat} is easily seen *not* to be locally fibrant; on the other hand, we may show that, for weak double categories \mathfrak{C} and \mathfrak{D} , if \mathfrak{D} is fibrant then so is $\mathfrak{Hom}(\mathfrak{C}, \mathfrak{D})$. It follows that the locally double bicategory \mathfrak{DbCat}_f , of fibrant double categories and all cells between them, is itself locally fibrant; and since any bicategory is trivially fibrant, that the locally double bicategory \mathfrak{Bicat} is too.

We will now show that every locally fibrant locally double bicategory gives rise to a tricategory. We begin with a technical result:

Proposition 20. *Let us write \mathbf{DbCat}_g for the maximal sub-2-category of \mathbf{DbCat}_f with only invertible 2-cells. Then the functor of mere categories $\mathbb{H}: \mathbf{DbCat} \rightarrow \mathbf{Bicat}$ can be extended to a trihomomorphism*

$$\mathbb{H}: \mathbf{DbCat}_g \rightarrow \mathbf{Bicat}.$$

Proof. First we define \mathbb{H} on cells. This is already done for 0- and 1-cells, and since \mathbf{DbCat}_g has no non-trivial 3-cells, it remains only to define it on 2-cells. So let there be given an invertible vertical transformation $\alpha: F \Rightarrow G: \mathfrak{C} \rightarrow \mathfrak{D}$. We define a pseudo-natural transformation $\mathbb{H}\alpha: \mathbb{H}F \Rightarrow \mathbb{H}G$ by taking

$$(\mathbb{H}\alpha)_x = \overline{\alpha_x}: Fx \rightarrow Gx, \quad \text{and} \quad (\mathbb{H}\alpha)_f = \begin{array}{ccc} Fx & \xrightarrow{Ff} & Fy \\ \downarrow \overline{\alpha_x} & \Downarrow \overline{\alpha_f} & \downarrow \overline{\alpha_y} \\ Gx & \xrightarrow{Gf} & Gy. \end{array}$$

The transformation axioms for $\mathbb{H}\alpha$ follow straightforwardly from the vertical transformation axioms for α and the functoriality of $(\overline{})$ with respect to 2-cell composition. Next we ensure that \mathbb{H} is locally a homomorphism of bicategories, which entails giving modifications $i_F: \text{id}_{\mathbb{H}F} \Rightarrow \mathbb{H}(\text{id}_F)$ and $m_{\alpha,\beta}: \mathbb{H}\beta \cdot \mathbb{H}\alpha \Rightarrow \mathbb{H}(\beta \cdot \alpha)$. These will have 2-cell components

$$(i_F)_x: \text{id}_{Fx} \Rightarrow \overline{\text{id}_{Fx}} \quad \text{and} \quad (m_{\alpha,\beta})_x: \overline{\beta_x} \cdot \overline{\alpha_x} \Rightarrow \overline{\beta_x \alpha_x}$$

in $\mathbb{H}\mathfrak{D}$ given by the pseudo-functoriality constraints for $(\overline{})$. The coherence axioms for these data therefore follow pointwise. Next, we must give adjoint pseudo-natural equivalences

$$\chi_{\mathfrak{C},\mathfrak{D},\mathfrak{C}}: \mathbb{H}(-) \otimes \mathbb{H}(?) \Rightarrow \mathbb{H}(- \otimes ?): \mathbf{DbICat}_g(\mathfrak{C}, \mathfrak{D}) \times \mathbf{DbICat}_g(\mathfrak{B}, \mathfrak{C}) \rightarrow \mathbf{Bicat}(\mathbb{H}\mathfrak{B}, \mathbb{H}\mathfrak{D}).$$

Observe that the homomorphisms $\mathbb{H}(-) \otimes \mathbb{H}(?)$ and $\mathbb{H}(- \otimes ?)$ agree on objects, and thus we may consider icons between them: in particular, any *invertible* icon between them will give rise to an adjoint pseudo-natural equivalence, and so to give χ it suffices to give invertible icons $\chi: \mathbb{H}(-) \otimes \mathbb{H}(?) \Rightarrow \mathbb{H}(- \otimes ?)$. So consider a pair of horizontally composable 2-cells

$$\begin{array}{ccccc} & F & & G & \\ & \curvearrowright & & \curvearrowright & \\ \mathfrak{B} & \Downarrow \alpha & \mathfrak{C} & \Downarrow \beta & \mathfrak{D} \\ & \curvearrowleft & & \curvearrowleft & \\ & F' & & G' & \end{array}$$

in \mathbf{DbICat}_g : we must give a modification $\chi_{\alpha,\beta}: \mathbb{H}\beta * \mathbb{H}\alpha \Rightarrow \mathbb{H}(\beta * \alpha)$. Now, these two pseudo-natural transformations have respective x -components given by

$$\begin{aligned} (\mathbb{H}\beta * \mathbb{H}\alpha)_x &= GFx \xrightarrow{\overline{\beta_{Fx}}} G'Fx \xrightarrow{G'\overline{\alpha_x}} G'F'x \\ \text{and } \mathbb{H}(\beta * \alpha)_x &= GFx \xrightarrow{\overline{G'\alpha_x \cdot \beta_{Fx}}} G'F'x, \end{aligned}$$

and so we take $(\chi_{\alpha,\beta})_x$ to be the 2-cell

$$G'\overline{\alpha_x} \cdot \overline{\beta_{Fx}} \xrightarrow{\mu_{\alpha_x} \cdot 1} \overline{G'\alpha_x} \cdot \overline{\beta_{Fx}} \xrightarrow{\cong} \overline{G'\alpha_x \cdot \beta_{Fx}}$$

of $\mathbb{H}\mathfrak{D}$. The modification axioms for $\chi_{\alpha,\beta}$ follow from the third fibrant homomorphism axiom and the functoriality axioms for $(\overline{})$ with respect to 2-cell composition. We must verify that these components $\chi_{\alpha,\beta}$ satisfy the three axioms making χ into an icon. The first is vacuous, whilst the second and third follow by a diagram chase using the axioms for a fibrant homomorphism. We argue entirely analogously in order to give the adjoint equivalences $\iota: I_{\mathbb{H}x} \Rightarrow \mathbb{H}(I_x)$.

Next we must give invertible modifications ω , δ and γ . In the case of ω , for

instance, this involves giving invertible modifications

$$\begin{array}{ccc}
& (\mathbb{H}(-) \otimes \mathbb{H}(?)) \otimes \mathbb{H}(*) & \\
\chi \otimes 1 \swarrow & & \searrow \mathfrak{a} \\
\mathbb{H}(- \otimes ?) \otimes \mathbb{H}(*) & & \mathbb{H}(-) \otimes (\mathbb{H}(?) \otimes \mathbb{H}(*)) \\
\downarrow \chi & \Rightarrow & \downarrow 1 \otimes \chi \\
\mathbb{H}((- \otimes ?) \otimes *) & & \mathbb{H}(-) \otimes \mathbb{H}(? \otimes *) \\
\searrow \mathbb{H}\mathfrak{a} & & \swarrow \chi \\
& \mathbb{H}(- \otimes (? \otimes *)) &
\end{array}$$

To do this, observe first that every pseudo-natural transformation bounding this diagram may also be viewed as an icon. We already know this for χ and hence also for $1 \otimes \chi$ and $\chi \otimes 1$; and it is so for \mathfrak{a} and $\mathbb{H}\mathfrak{a}$ since composition of 1-cells in both \mathbf{DbICat}_g and \mathbf{Bicat} is *strictly* associative. If we now compose all the 2-arrows in this diagram *qua* icons, we obtain two further icons $\sigma, \tau: (\mathbb{H}(-) \otimes \mathbb{H}(?)) \otimes \mathbb{H}(*) \Rightarrow \mathbb{H}(- \otimes (? \otimes *))$: and a long but straightforward diagram chase with the fibrant homomorphism axioms shows that these two icons are, in fact, equal.

On the other hand, if we compose the two sides *qua* pseudo-natural transformations, then the pseudo-naturals that we get will not necessarily be icons, but they will, at least, be *isomorphic* to icons, namely the icons σ and τ respectively. Thus we take ω to be the composite of the invertible modification from the left-hand side of this diagram to $\sigma = \tau$ and the invertible modification from τ to the right-hand side. We proceed similarly for the invertible modifications δ and γ .

The final thing to check are the two trihomomorphism axioms, equating certain pastings of 3-cells in \mathbf{Bicat} . But all the 3-cells in question are either coherence 3-cells of \mathbf{Bicat} ; or component 3-cells of ω, δ and γ . But these latter 3-cells are in turn built from coherence 3-cells of \mathbf{Bicat} and coherence 3-cells for the local homomorphisms $(\overline{})$. The result thus follows by coherence for tricategories and bicategorical coherence for functors. \square

Theorem 21. *Let \mathfrak{C} be a locally fibrant locally double bicategory with chosen companions in each hom. Then there is a tricategory \mathcal{T} with the same objects as \mathfrak{C} , and*

$$\mathcal{T}(A, B) = \mathbb{H}(\mathfrak{C}(A, B)).$$

Proof. We begin by observing that both \mathbf{DbICat}_g and \mathbf{Bicat} come equipped with finite product structure; and that the trihomomorphism \mathbb{H} preserves the cartesian product of j -cells for $j = 0, 1, 2, 3$. Now, the top-level composition and identity functors for \mathcal{T} are given by applying \mathbb{H} to the corresponding data (LDD3) and (LDD4) for \mathfrak{C} :

$$1 = \mathbb{H}1 \xrightarrow{\mathbb{H} \lfloor I_A \rfloor} \mathbb{H}(\mathfrak{C}(A, A)) = \mathcal{T}(A, A)$$

and

$$\mathcal{T}(B, C) \times \mathcal{T}(A, B) = \mathbb{H}(\mathfrak{C}(B, C) \times \mathfrak{C}(A, B)) \xrightarrow{\mathbb{H} \otimes} \mathbb{H}(\mathfrak{C}(A, C)) = \mathcal{T}(A, C).$$

To obtain the pseudo-natural adjoint equivalences \mathfrak{a} , \mathfrak{l} and \mathfrak{r} witnessing the associativity and unitality of this composition, we apply \mathbb{H} to the corresponding data (LDD5) and (LDD6) for \mathfrak{C} . Since each of a , l and r is an adjoint equivalence (in fact, an isomorphism) in the relevant hom of \mathbf{DbICat}_g , the same will obtain for their images in \mathbf{Bicat} ; and because \mathbb{H} strictly preserves both cartesian products and composition of 1-cells, these adjoint equivalences will have the correct sources and targets.

Next we must give the invertible modifications π , μ , λ and ρ . To obtain π , for example, we begin by applying \mathbb{H} to the axiom (LDA2) for \mathfrak{C} . This yields an equality of 2-cells in \mathbf{Bicat} ; however, these 2-cells are not of the right form to be the source and target of π . In order to make them so, we may adjust by coherence 3-cells in \mathbf{Bicat} whose existence is guaranteed by the coherence theorem for trihomomorphisms. Consequently, we may take π to be given by the composite of these coherence 3-cells; and similarly for μ , λ and ρ .

Finally, we must check the three tricategory axioms. These are normally stated in a “local” form, asserting the equality of certain pastings of 3-cells in the relevant hom-bicategories; but in this situation, it will be more appropriate to consider them in their “global” form. Each such axiom amounts to giving a diagram of 2- and 3-cells in \mathbf{Bicat} , whose vertices are pasting diagrams built from copies of the 2-cells a , l and r , and whose arrows are 3-cells between those 2-cells, built from copies of π , μ , λ and ρ ; and asserting that the two ways around this diagram coincide.

To show this, we consider the corresponding diagram for \mathfrak{C} . This is a diagram of 2- and 3-cells in \mathbf{DbICat}_g , which since \mathbf{DbICat}_g has only identity 3-cells, must commute. Hence by applying \mathbb{H} we obtain a commutative diagram in \mathbf{Bicat} , which, unfortunately, has both the wrong vertices and the wrong arrows. Nonetheless, by the coherence theorem for functors, each “wrong” vertex admits an isomorphism 3-cell to the “right” vertex; and in such a way that composing these isomorphism 3-cells with the “wrong” arrows yields the “right” arrows. \square

Special cases of this theorem give us new proofs of some existing results. Restricting to the one-object case, we have the result that *any fibrant monoidal double category gives rise to a monoidal bicategory*; this statement and a sketch proof appear as Theorem B.4 of [17]. In particular, we obtain elegant proofs that the bicategories of rings and bimodules, of categories and profunctors, and of spans internal to a cartesian category \mathcal{C} are all monoidal bicategories.⁶ Finally, applying this theorem to the fibrant locally double bicategory \mathfrak{Bicat} , we deduce the existence of a tricategory of bicategories \mathbf{Bicat} . Again, the result is not new, but the proof is, showing how the tricategory structure on \mathbf{Bicat} may be induced from a piece of canonical, universally determined structure, namely the biclosed structure on \mathbf{DbICat} .

⁶For the last two of these examples, the machinery of [2] provides another elegant proof of this fact, and in fact goes further, showing that the monoidal bicategories in question are *symmetric* monoidal bicategories.

6 A tricategory of tricategories

We now wish to apply Theorem 21 to the locally double bicategory \mathfrak{Tricat}_3 , and in order to do this, we must first prove its homs to be fibrant.

Proposition 22. *Each weak double category $\mathfrak{Icon}(\mathcal{S}, \mathcal{T})$ is fibrant.*

Proof. Suppose we are given an invertible ico-icon $\alpha: F \Rightarrow G: \mathcal{S} \rightarrow \mathcal{T}$. We must provide a pseudo-icon $\bar{\alpha}: F \Rrightarrow G$ and an invertible icon modification

$$\begin{array}{ccc} F & \xRightarrow{\bar{\alpha}} & G \\ \alpha \downarrow & \Downarrow \epsilon_\alpha & \downarrow \text{id}_G \\ G & \xRightarrow{I_G} & G. \end{array}$$

Now by Proposition 11, there is a bijection between the ico-icons $F \Rightarrow G$ and the oplax icons $F \Rrightarrow G$ with identity 2-cell components: for which the invertible ico-icons on the one side correspond to the pseudo-icons on the other. Thus we take $\bar{\alpha}$ to be the pseudo-icon corresponding to α under this bijection.

To give the icon modification ϵ_α , we must give 3-cells $(\epsilon_\alpha)_f: \bar{\alpha}_f \Rightarrow (I_G)_f$ of \mathcal{T} , forming the components of an $\text{ob } \mathcal{S} \times \text{ob } \mathcal{S}$ -indexed family of cubical modifications, and satisfying axioms (MA1) and (MA2). Since we have $\bar{\alpha}_f = (I_G)_f = \text{id}_{Ff}: Ff \Rightarrow Ff$, we take $(\epsilon_\alpha)_f = \text{id}_{\text{id}_f}$. The cubical modification axioms and axioms (MA1) and (MA2) now follow by coherence for bicategories. \square

Thus \mathfrak{Tricat}_3 is locally fibrant, and so we may apply Theorem 21 to it, and deduce:

Theorem 23. *There is a tricategory \mathbf{Tricat}_3 with objects being tricategories; 1-cells, lax homomorphisms; 2-cells, pseudo-icons; and 3-cells, globular icon modifications.*

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